Superfluid helicity and vortex reconnections

L. Sriramkumar
Racah Institute of Physics, Hebrew University,
Givat Ram, Jerusalem 91904, Israel *.

Abstract

Vortex reconnections are considered to be an essential mechanism that sustains the chaotic state of a vortex tangle in turbulent helium II. In a pure turbulent superflow of helium II, superfluid helicity is a topologically invariant quantity that counts the extent of linkage of closed vortex knots and loops. This implies that superfluid helicity will *not* be conserved whenever vortex reconnections take place. However, I find that, within the two-fluid model, superfluid helicity *is* a conserved quantity even when the restoring and drag forces (that arise due to the interaction of the superfluid vortices with the normal fluid) are taken into account. I briefly discuss the implications of this result.

PACS: 67.40.-w; 67.40.Bz; 67.40.Vs

Keywords: Superfluid turbulence; Helicity; Vortex reconnections

*Present address: Theoretical Physics Institute, Department of Physics, University of Alberta, Edmonton, Alberta T6G 2J1, Canada. E-mail: slakshm@phys.ualberta.ca.

I. SUPERFLUID TURBULENCE AND HELICITY

Superfluid ^{4}He, conventionally referred to as helium II, behaves as an irrotational ideal fluid which can contain quantized vortices. In addition, it supports a gas of elementary excitations, viz. the normal fluid, which exerts a frictional force on the quantized vortices. Such a hydrodynamical description of helium II, called the two-fluid model, is valid down to scales comparable to the coherence length of the superfluid condensate, a distance which both theory and experiment suggest to be of the order of a few Angstroms (see, for e.g., Refs. [1,2]).

Helium II exhibits turbulence as ordinary fluids do [3]. The turbulent superfluid is considered to consist of a random tangle of quantized vortices. However, due to the fact that circulation is quantized, unlike ordinary fluids, helium II exhibits turbulence only above a certain critical velocity. Moreover, as helium II consists of two fluids, different types of turbulent states are possible depending on the relative velocity between the normal and the superfluid components (see Ref. [3]; also see Ref. [2], Chap. 7).

An important aspect of turbulence in helium II are vortex reconnections. It was recognized very early in literature that topology-changing vortex reconnections are an essential mechanism that sustains the chaotic state of the vortex tangle [4]. Despite that fact, there still does not exist an adequate picture of vortex reconnections. (This is indeed the case at least within the two-fluid model. However, it should be mentioned here that vortex reconnections have been shown to occur when the condensate wave function is assumed to evolve according to the Gross-Pitaevskii equation [5].) The most successful approach that has been developed to describe the homogeneous turbulence in helium II is the model due to Schwarz (see Ref. [6] and the earlier references to Schwarz therein). Using his model, Schwarz has been able to numerically simulate the steady-state density of quantized vortices that occur in homogeneous turbulent states. However, even in the Schwarz's model, vortex reconnection was only proposed as an *Ansatz*.

In the case of turbulence in ordinary fluids, initially Moreau [7] and, independently later, Moffatt [8,9] introduced helicity as a measure of the degree of tangledness of closed vortex knots and loops. The helicity $\mathcal{H}$ corresponding to a velocity field $\mathbf{v}$ of the fluid is defined as:

$$\mathcal{H} = \int_{\mathcal{V}} d^3x \; [\mathbf{v}(\mathbf{x}) \cdot \boldsymbol{\omega}(\mathbf{x})] \, , \tag{1}$$

where $\boldsymbol{\omega} = (\boldsymbol{\nabla} \times \mathbf{v})$ is the vorticity field and $\mathcal{V}$ is the volume of the fluid containing the vortex tangle. Helicity $\mathcal{H}$ as defined above is a pseudoscalar and is, in general, expected to have a non-zero value in a turbulent flow. It has been pointed out earlier that helicity can prove to be a useful measure to describe turbulence in non-linear fields or order parameters such as the condensate wave function describing superfluid ^{4}He (see Refs. [10,11]; also see Refs. [12,13] in this context). My aim in this Letter is to examine how the concept of helicity can possibly be utilized in an attempt to understand the mechanism of vortex reconnections in turbulent helium II.

II. SUPERFLUID HELICITY AS A LINKING NUMBER

Amongst the different possible turbulent states of helium II, the state I shall be interested in is a pure turbulent superflow, a state wherein the normal fluid is at rest [14]. Interestingly enough, unlike the other turbulent states of helium II, turbulence in a pure superflow is independent of the geometry of the flow tube and, in fact, a homogeneous turbulent state occurs in all geometries [15,16].

In a pure turbulent superflow, since the normal fluid is at rest, helicity for a given volume $\mathcal{V}$ of helium II can be defined as

$$\mathcal{H}_{\rm sf} = \int_{\mathcal{V}} d^3x \, \left[\mathbf{v}_{\rm s}(\mathbf{x}) \cdot \boldsymbol{\omega}_{\rm s}(\mathbf{x})\right], \tag{2}$$

where $\mathbf{v}_{\rm s}$ is the velocity of the superfluid component and $\boldsymbol{\omega}_{\rm s}$ is the vorticity associated with the superflow. Given a vorticity field $\boldsymbol{\omega}_{\rm s}$ that is confined to a finite volume, the corresponding superfluid velocity field $\mathbf{v}_{\rm s}$ can be expressed using the Biot-Savart law as follows (see, for instance, Refs. [17,18]):

$$\mathbf{v}_{\rm s}(\mathbf{x}) = \int d^3x' \left[\frac{\boldsymbol{\omega}_{\rm s}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}\right]. \tag{3}$$

On substituting this expression for $\mathbf{v}_{\rm s}$ in the definition (2) for the superfluid helicity, I obtain that

$$\mathcal{H}_{\rm sf} = \int_{\mathcal{V}} d^3x \int_{\mathcal{V}} d^3x' \left[\frac{\boldsymbol{\omega}_{\rm s}(\mathbf{x}) \cdot \boldsymbol{\omega}_{\rm s}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}\right]. \tag{4}$$

In helium II, the vorticity field $\boldsymbol{\omega}_{\rm s}$ is confined to the superfluid vortices which are extremely thin with a core radius typically of the order of a few Angstroms (see, for e.g., Ref. [1], pp. 205–206; also see Ref. [19]). Even though the superfluid vortices are extremely thin, the contribution to helicity due to the non-zero thickness of the vortices cannot be ignored. Due to this reason, for the discussion that follows immediately, I shall consider a superfluid vortex with a circulation κ to be composed of infinitesimally thin vortex *lines* with circulation $\bar{\kappa}$. (I will do so despite the fact that the circulation of the superfluid vortices is quantized. I will revert to discussion in terms of the quantized circulation κ soon after.) Then, in the expression (4) for the superfluid helicity $\mathcal{H}_{\rm sf}$, I can write

$$\int d^3x \; \boldsymbol{\omega}_{\rm s}(\mathbf{x}) = \int d\mathbf{s} \int d^2x_{\perp} \; \omega_{\rm s}(\mathbf{x}) \equiv \sum_i \bar{\kappa}_i \int d\bar{\mathbf{s}}_i, \tag{5}$$

where $d\bar{\mathbf{s}}_i$ and $\bar{\kappa}_i$ denote the infinitesimal element and the circulation associated with the vortex line i and the sum extends over all the vortex lines within a given vortex.

I shall now assume that the superfluid vortex tangle within the volume $\mathcal{V}$ contains *only* closed vortex knots and loops. In such a case, the expression (4) for the superfluid helicity can be *formally* written as

$$\mathcal{H}_{\rm sf} = (4\pi) \sum_{i,j} \left(\bar{\mathcal{L}}_{ij} \, \bar{\kappa}_i \, \bar{\kappa}_j\right), \tag{6}$$

where $\bar{\kappa}_i$ is the circulation associated with a given closed vortex *line* ℓ_i and $\bar{\mathcal{L}}_{ij}$ denotes the following Gauss integral along any two closed vortex lines ℓ_i and ℓ_j [20]:

$$\bar{\mathcal{L}}_{ij} = \left(\frac{1}{4\pi}\right) \oint_{\ell_i} \oint_{\ell_j} \left[\frac{d\mathbf{s} \cdot d\mathbf{s}' \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}\right]. \tag{7}$$

(It should be noted here that $\bar{\mathcal{L}}_{ij}$ is a topologically invariant quantity *only* when $i \neq j$ and it is not when $i = j$. I shall comment further on this point in the next paragraph.) The sum in the expression (6) extends over *all* the closed vortex *lines* and care should be exercised in rewriting this sum as a sum over the tubular vortex knots and loops. Due to the fact that the superfluid vortices have a non-zero thickness, in addition to the contribution to helicity when ℓ_i and ℓ_j are considered to be vortex lines in two *different* vortex tubes (say, a and b), one need to take into account the contribution when ℓ_i and ℓ_j are considered to be two *different* lines within the *same* vortex tube (say, a). The former would then correspond to the topologically invariant Gauss linking number $\mathcal{L}_{ab}$ of the two different vortices a and b and the latter would correspond to the linking number of the two different vortex lines in the same vortex a, a quantity that I shall refer to as the self-linking number $\mathcal{S}l_a$ of the vortex, which is topologically invariant as well (cf. Refs. [21,22]; see Ref. [23] for a recent discussion). Therefore, the superfluid helicity $\mathcal{H}_{\mathbf{sf}}$ can be written as

$$\mathcal{H}_{\mathbf{sf}} = (4\pi) \sum_{a,b}^{a \neq b} (\mathcal{L}_{ab}\, \kappa_a\, \kappa_b) + (4\pi) \sum_a \left(\mathcal{S}l_a\, \kappa_a^2\right), \tag{8}$$

where the sum in the second term extends over *all* the vortices, whereas the sum in the first term extends over all *pairs* of vortices confined to the volume $\mathcal{V}$.

The following comments and clarifications are in order at this stage of the discussion. I would like to stress again that the self-linking number $\mathcal{S}l_a$ of a vortex a is the Gauss linking number of two *different* vortex *lines* (say, ℓ_i and ℓ_j with $i \neq j$) within the *same* vortex. Now, using Calugareanu's theorem[1], the self-linking number $\mathcal{S}l_a$ of a particular vortex a can be decomposed as follows:

$$\mathcal{S}l_a = \mathcal{W}r_a + \mathcal{T}w_a, \tag{9}$$

where $\mathcal{W}r_a$ is a quantity referred to as the writhe and $\mathcal{T}w$ is called the twist of the vortex. The twist $\mathcal{T}w_a$ reflects the extent to which a given vortex line (say, ℓ_j) "twists" around another line (say, ℓ_i) within the same vortex a and the quantity writhe $\mathcal{W}r_a$ corresponds to the case wherein $i = j$ in $\bar{\mathcal{L}}_{ij}$ (i.e. when $\ell_i = \ell_j$ within the same vortex a). (For an exact definition of the twist, see, for instance, Ref. [23].) Had I considered the superfluid vortices to be infinitesimally thin and had, therefore, treated the sum over the vortex lines in Eq. (6)

[1]This theorem is usually referred to in literature as the White-Fuller relation [24,25]. However, as Moffatt and Ricca point out [23], Calugareanu had, in fact, obtained this relation almost a decade before White and Fuller. Following Moffatt and Ricca, I shall refer to this relation as Calugareanu's theorem.

as the sum over the superfluid vortices themselves, I would have taken into account the contribution to superfluid helicity due to the writhe $\mathcal{W}r_a$, but would have missed out the contribution due to the twist $\mathcal{T}w_a$. (A twisted vortex, for instance, will have a higher energy than one that is not and, hence, it should be, in principle, possible to distinguish between these two vortices; see, for e.g., Ref. [26] in this context.) Moreover, as I have pointed out above, the writhe $\mathcal{W}r_a$, when it stands alone, is *not* a topologically invariant quantity. It is *only* its sum with the twist $\mathcal{T}w_a$, viz. the self-linking number $\mathcal{S}l_a$, that is topologically invariant (cf. Ref. [23]). Therefore, had I ignored the non-zero thickness of the superfluid vortices, I would have obtained an expression for the superfluid helicity that would not be topologically invariant.

The circulation κ is related to the winding number n of the superfluid vortex by the relation (cf. Ref. [1], p. 182): $\kappa = (2\pi n\hbar/m)$, where m is the mass of the ^{4}He atom. Superfluid vortices with winding number $n > 1$ are known to be unstable (see, for instance, Ref. [17]) and, hence, one can expect that the turbulent superfluid predominantly contains vortices with unit winding number. Then, the helicity of the superfluid vortex tangle as given by Eq. (8) reduces to

$$\mathcal{H}_{\rm sf} = (4\pi)\,(2\pi\hbar/m)^2\,\left(\sum_{a,b}^{a\neq b}\mathcal{L}_{ab} + \sum_a \mathcal{S}l_a\right). \tag{10}$$

Evidently, in a pure turbulent superflow of helium II, superfluid helicity $\mathcal{H}_{\rm sf}$ is a topologically invariant quantity that counts the extent of linkage of closed superfluid vortex knots and loops.

III. CONSERVATION OF SUPERFLUID HELICITY

In the two-fluid model, when the vorticity associated with the superflow is non-zero, the superfluid velocity $\mathbf{v}_{\rm s}$ satisfies the following equation of motion (cf. Ref. [27]; also see Refs. [28,29]):

$$\left(\frac{D_{\rm s}\mathbf{v}_{\rm s}}{Dt}\right) = \left(-\boldsymbol{\nabla}\mu + \mathbf{F}_{\rm r} + \mathbf{F}_{\rm ns}\right), \tag{11}$$

where μ is a scalar function that denotes the chemical potential, $\mathbf{F}_{\rm r}$ is a restoring force that tends to straighten curved vortices and $\mathbf{F}_{\rm ns}$ is the mutual frictional force that arises due to the interaction of the normal fluid with the superfluid vortices. The differential operator $(D_{\rm s}/Dt)$ appearing in the above equation of motion is defined as

$$\frac{D_{\rm s}}{Dt} \equiv \frac{\partial}{\partial t} + (\mathbf{v}_{\rm s}\cdot\boldsymbol{\nabla}). \tag{12}$$

In the case of turbulence in a pure superflow, since the normal fluid is at rest, the gradient of the chemical potential μ is given by

$$\boldsymbol{\nabla}\mu = \left(\frac{1}{\rho}\right)\boldsymbol{\nabla}p - S\,\boldsymbol{\nabla}T - \left(\frac{\rho_{\rm n}}{2\rho}\right)\boldsymbol{\nabla}\left(|\mathbf{v}_{\rm s}|^2\right) + \left(\frac{\lambda}{\rho}\right)\boldsymbol{\nabla}\left(|\boldsymbol{\omega}_{\rm s}|\right), \tag{13}$$

where $\rho = (\rho_{\rm s} + \rho_{\rm n})$, $\rho_{\rm n}$ and $\rho_{\rm s}$ are the normal and the superfluid densities, p denotes the pressure, S the entropy per unit mass, T the temperature and the quantity λ is a phenomenological function which relates the increase in the energy density that results from an increase in the number of superfluid vortices. Also, it has been assumed that, as in the case of ordinary fluids, the superfluid vorticity $\boldsymbol{\omega}_{\rm s}$ is defined as

$$\boldsymbol{\omega}_{\rm s} = (\boldsymbol{\nabla} \times \mathbf{v}_{\rm s}) . \tag{14}$$

The restoring force $\mathbf{F}_{\rm r}$ is given by the expression

$$\mathbf{F}_{\rm r} = -\Big(\boldsymbol{\omega}_{\rm s} \times \mathbf{F}\Big), \tag{15}$$

where

$$\mathbf{F} = \left(\frac{1}{\rho_{\rm s}}\right) \Big(\boldsymbol{\nabla} \times (\lambda\, \hat{\omega}_{\rm s})\Big) \qquad \text{and} \qquad \hat{\omega}_{\rm s} = (\boldsymbol{\omega}_{\rm s}/|\boldsymbol{\omega}_{\rm s}|) . \tag{16}$$

The mutual frictional force $\mathbf{F}_{\rm ns}$ is given by the expression

$$\begin{aligned} \mathbf{F}_{\rm ns} = \left(\frac{B'\rho_{\rm n}}{2\rho}\right) \Big[\boldsymbol{\omega}_{\rm s} \times (\mathbf{v}_{\rm s} + \mathbf{F})\Big] + \left(\frac{B\rho_{\rm n}}{2\rho}\right) \Big(\hat{\omega}_{\rm s} \times \Big[\boldsymbol{\omega}_{\rm s} \times (\mathbf{v}_{\rm s} + \mathbf{F})\Big]\Big) \\ - (\gamma\rho_{\rm s}) \Big(\hat{\omega}_{\rm s} \Big[\boldsymbol{\omega}_{\rm s} \cdot (\mathbf{v}_{\rm s} + \mathbf{F})\Big]\Big) , \end{aligned} \tag{17}$$

where B and B' are the Hall and Vinen coffecients. The last term involving the coefficient γ represents a longitudinal mutual frictional force which is very small compared to B and B' and hence can be neglected.

I had pointed out in the last section that the superfluid vortices are extremely thin and, hence, the vorticity field $\boldsymbol{\omega}_{\rm s}$ is confined to these thin vortices. By assuming that the superfluid vorticity is given by the expression (14), the vorticity associated with these thin vortices has in effect been smoothed over a finite extent. The equation of motion for the vorticity field $\boldsymbol{\omega}_{\rm s}$ can now be obtained from the equation of motion (11) for the superfluid velocity field. It is given by

$$\left(\frac{D_{\rm s}\boldsymbol{\omega}_{\rm s}}{Dt}\right) = [(\boldsymbol{\omega}_{\rm s} \cdot \boldsymbol{\nabla})\, \mathbf{v}_{\rm s} - (\boldsymbol{\nabla} \cdot \mathbf{v}_{\rm s})\, \boldsymbol{\omega}_{\rm s}] + \boldsymbol{\nabla} \times (\mathbf{F}_{\rm r} + \mathbf{F}_{\rm ns}) . \tag{18}$$

Since the normal fluid is at rest in a pure turbulent superflow, I can (to a good approximation) assume that its density is a constant. Then, the equation of continuity of mass reduces to

$$\left(\frac{\partial \rho_{\rm s}}{\partial t}\right) + \boldsymbol{\nabla} \cdot (\rho_{\rm s} \mathbf{v}_{\rm s}) = 0. \tag{19}$$

If I now assume that the volume $\mathcal{V}$ containing the superfluid vortex tangle is moving with the superflow, then

$$\frac{D_{\rm s}}{Dt} \Big(\rho_{\rm s}\, d^3\mathbf{x}\Big) = 0. \tag{20}$$

From the equation of motion (18) for the vorticity field $\boldsymbol{\omega}_\mathrm{s}$ and the continuity equation (19), I obtain that

$$\frac{D_\mathrm{s}}{Dt}\left(\frac{\boldsymbol{\omega}_\mathrm{s}}{\rho_\mathrm{s}}\right) = \left(\frac{1}{\rho_\mathrm{s}}\right)\Big[(\boldsymbol{\omega}_\mathrm{s}\cdot\boldsymbol{\nabla})\,\mathbf{v}_\mathrm{s} + \boldsymbol{\nabla}\times(\mathbf{F}_\mathrm{r}+\mathbf{F}_\mathrm{ns})\Big]. \tag{21}$$

Using this equation and Eqs. (11) and (20), it is then easy to show that

$$\left(\frac{D_\mathrm{s}\mathcal{H}_\mathrm{sf}}{Dt}\right) = \int_{\mathcal{V}} d^3x\,\left(\boldsymbol{\nabla}\cdot\Big[\Big(|\mathbf{v}_\mathrm{s}|^2/2\Big)-\mu\Big]\,\boldsymbol{\omega}_\mathrm{s}\right)$$
$$-\int_{\mathcal{V}} d^3x\,\Big(\boldsymbol{\nabla}\cdot[\mathbf{v}_\mathrm{s}\times(\mathbf{F}_\mathrm{r}+\mathbf{F}_\mathrm{ns})]\Big) + 2\int_{\mathcal{V}} d^3x\,\Big[\boldsymbol{\omega}_\mathrm{s}\cdot(\mathbf{F}_\mathrm{r}+\mathbf{F}_\mathrm{ns})\Big]. \tag{22}$$

Since the restoring force $\mathbf{F}_\mathrm{r}$ and the mutual frictional force $\mathbf{F}_\mathrm{ns}$ are perpendicular to the vorticity field $\boldsymbol{\omega}_\mathrm{s}$ (see Eqs. (15) and (17) above), the last term in the above expression vanishes. The remaining two terms can be rewritten using Gauss' divergence theorem as follows

$$\left(\frac{D_\mathrm{s}\mathcal{H}_\mathrm{sf}}{Dt}\right) = \int_{\mathcal{S}} d^2x\,\left(\hat{n}\cdot\boldsymbol{\mathcal{H}}_\mathrm{sf}\right), \tag{23}$$

where $\mathcal{S}$ is the surface enclosing the volume $\mathcal{V}$ and $\hat{n}$ is the unit-vector normal to the surface $\mathcal{S}$. The quantity $\boldsymbol{\mathcal{H}}_\mathrm{sf}$ is the current associated with the superfluid helicity and is given by the expression

$$\boldsymbol{\mathcal{H}}_\mathrm{sf} = \left(\Big[\Big(|\mathbf{v}_\mathrm{s}|^2/2\Big)-\mu\Big]\,\boldsymbol{\omega}_\mathrm{s} - [\mathbf{v}_\mathrm{s}\times(\mathbf{F}_\mathrm{r}+\mathbf{F}_\mathrm{ns})]\right). \tag{24}$$

Evidently, the superfluid helicity $\mathcal{H}_\mathrm{sf}$ is a conserved quantity [10].

In the last section, I had shown that the superfluid helicity $\mathcal{H}_\mathrm{sf}$ measures the extent of linkage of closed vortex knots and loops in a vortex tangle. The fact that superfluid helicity is conserved clearly implies that vortex reconnections are precluded within the two-fluid model even when the restoring and drag forces are taken into account. It should be mentioned here that, in turbulent states wherein the normal fluid is in motion, the total helicity of helium II will contain a cross term involving the normal fluid velocity $\mathbf{v}_\mathrm{n}$ and the superfluid vorticity $\boldsymbol{\omega}_\mathrm{s}$. Hence, when the normal fluid is in motion, the helicity for a given volume of helium II will not reflect the extent of linkage of closed vortex knots and loops within that volume. It is for this reason that I had confined my discussion to turbulence in a pure superflow.

IV. DISCUSSION

In a classical ideal fluid, Helmholtz's theorem ensures that the vortex lines move with the fluid and hence no vortex reconnections occur. But, in turbulent helium II, due to the presence of the restoring and drag forces $\mathbf{F}_\mathrm{r}$ and $\mathbf{F}_\mathrm{ns}$, the superfluid vortices, in general, do not move with the superflow (see Ref. [28], Eq. (16-45) in this context). However, as I have shown, helicity is conserved even when these forces are taken into account. Clearly, the fact that the vortices cease to move with the flow does not necessarily imply that vortex reconnections will take place—the forces $\mathbf{F}_{\mathrm{r}}$ and $\mathbf{F}_{\mathrm{ns}}$ stretch and drag and the superfluid vortex knots and loops in such a fashion that vortex reconnections do not occur.

However, it is important to realize that the fact that helicity is conserved within the two-fluid model does not imply vortex reconnections will not take place in reality. (As I have mentioned before, vortex reconnections are considered to be essential to sustain the chaotic state of the vortex tangle Ref.schwarz88.) In fact, the equations of the two-fluid model, at least as they presently stand, can be expected to breakdown when the vortices approach each other. In classical fluids, it is known that when viscous forces are present, vortices can move towards each other and eventually reconnect [30,31]. This suggests that additional forces, possibly viscous in nature, may need to be introduced, if vortex reconnections are to be accounted for within the two-fluid model [10].

In spite of such "improvements" that can possibly be made to the two-fluid model, the quantum nature of the vortices cannot be ignored when the vortices approach within a few core lengths of each other [32]. When considering the fact that the superfluid helicity of a vortex tangle containing closed vortex knots and loops is proportional to an integer, it is tempting to propose the following quantum picture of vortex reconnections. The superfluid helicity (or, equivalently, the total linking number) of a vortex tangle can be considered as a quantum number describing the stationary and turbulent states of the condensate wave function of ^{4}He. In such a picture, vortex reconnections can be interpreted as a quantum transition between the stationary states of the condensate wave function described by different helicity quantum numbers.

ACKNOWLEDGMENTS

I would like to thank Prof. Jacob Bekenstein for suggesting the problem and for discussions during the course of this work. I would also wish to thank Tsippora Mendelson and Profs. William Glaberson, Yu. G. Mamaladze, Don Page and Terry Gannon for discussions, Prof. Brandon Carter for pointing out an error in the previous version of this manuscript and the anonymous referee for bringing Ref. [10] to my attention. This work was supported by the Israel Science Foundation established by the Israel Academy of Sciences and by the National Science and Engineering Research Council, Canada.

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