Lecture 1: Linear Vector Space

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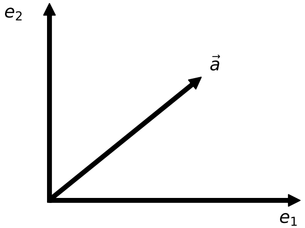
Vectors

Important objects having both magnitude and direction



Eg., momentum $\vec{p} = m\vec{v}$, force $\vec{f} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}$

- To generalize to include vectors with complex components

  Dirac notation, $\underbrace{\vec{a}}_{\text{vector}} \equiv \underbrace{|a\rangle}_{\text{ket}}$ is most convenient

- The set of kets $\{|a\rangle, |b\rangle, |c\rangle \dots\} \xrightarrow{\text{denoted as}} \mathcal{K}$ (ket space)

Ket Space $\mathcal{K}$

It is a linear vector space of $\{|a\rangle, |b\rangle, |c\rangle \ldots\}$ that is closed under

Addition

$$\begin{aligned} |a\rangle + |b\rangle &= |c\rangle = |b\rangle + |a\rangle \\ (|a\rangle + |b\rangle) + |c\rangle &= |a\rangle + (|b\rangle + |c\rangle) \end{aligned}$$

Multiplication

$$\begin{aligned} \lambda(|a\rangle + |b\rangle) &= \lambda |a\rangle + \lambda |b\rangle \\ (\lambda + \mu) |a\rangle &= \lambda |a\rangle + \mu |a\rangle \\ \lambda(\mu |a\rangle) &= (\lambda\mu) |a\rangle \end{aligned}$$

Linear Independence

- The set $\mathbb{B} = \{|e_1\rangle, |e_2\rangle, \ldots |e_N\rangle\}$ is linearly independent, if

$$\sum_i \lambda_i |e_i\rangle = 0, \quad \text{with} \quad \lambda_i = 0$$

- $\mathcal{K}$ is N-dimensional, if no more than N such kets exist
- An arbitrary ket in $\mathcal{K}$ is specified using $\mathbb{B}$,

$$|a\rangle = \sum_i a_i |e_i\rangle, \quad a_i \in \mathbb{C}$$

- $\mathbb{B}$ is thus called a **basis** and $|e_i\rangle$ are called **base kets**

Bra Space $\mathcal{B}$

- It is **dual** to the $\mathcal{K}$ space
- Typical bras in $\mathcal{B}$ are denoted as $\langle a| , \langle b| , \langle c| \ldots$
- $\mathcal{B}$ is therefore spanned by some basis $\{\langle e_1| , \langle e_2| , \ldots \langle e_N|\}$

Dual Correspondence (DC)

$$
\begin{aligned}
|a\rangle &\overset{\text{DC}}{\longleftrightarrow} \langle a| \\
\{|e_1\rangle , |e_2\rangle , \ldots |e_N\rangle\} &\overset{\text{DC}}{\longleftrightarrow} \{\langle e_1| , \langle e_2| , \ldots \langle e_N|\} \\
|a\rangle + |b\rangle &\overset{\text{DC}}{\longleftrightarrow} \langle a| + \langle b| \\
c_a |a\rangle + c_b |b\rangle &\overset{\text{DC}}{\longleftrightarrow} c_a^* \langle a| + c_b^* \langle b|
\end{aligned}
$$

Vector Products

Two useful products exist,

- Inner product: $(\langle a|) \cdot (|b\rangle) = \langle a|b\rangle$ "scalar"
- Outer product: $(|a\rangle) \cdot (\langle b|) = |a\rangle \langle b|$ "operator"
- Illegal product: $(|a\rangle) \cdot (|b\rangle) = |a\rangle |b\rangle$ "meaningless"

Postulates for Inner Product

$$\langle a|b\rangle = \langle b|a\rangle^* \quad \text{"non-commutative"}$$
$$\langle a|a\rangle \geq 0 \quad \text{"norm is positive-definite"}$$

Non null kets can be always normalized by

$$\frac{|a\rangle}{\langle a|a\rangle^{1/2}} \longrightarrow |a\rangle_{\text{normalized}}$$

Illustration of Vector Products

Consider a standard (ortho-normal) basis,

$$\mathbb{B} = \{|e_1\rangle, |e_2\rangle, \ldots |e_N\rangle\}, \quad \langle e_i|e_j\rangle = \delta_{ij}$$

A random ket in $\mathbb{B}$ can be written as,

$$|a\rangle = \sum_i a_i |e_i\rangle, \quad a_i \in \mathbb{C}$$

The inner product is a scalar

$$\langle a|a\rangle = \underbrace{\left(\sum_i a_i^* \langle e_i|\right)}_{\langle a|} \underbrace{\left(\sum_j a_j |e_j\rangle\right)}_{|a\rangle} = \sum_i |a_i|^2 \geq 0.$$

The outer product is a projection operator

$$\underbrace{|e_j\rangle\langle e_j|}_{\text{operator}} \underbrace{|a\rangle}_{\text{ket}} = |e_j\rangle\langle e_j| \sum_i a_i |e_i\rangle = a_j |e_j\rangle$$

Homework

Prove the following

- **Pythagoras theorem:** If $|a\rangle$ and $|b\rangle$ are orthogonal, then

$$\left|\,|a\rangle + |b\rangle\,\right|^2 = \langle a|a\rangle + \langle b|b\rangle$$

- **Parallelogram law:** For arbitrary $|a\rangle$ and $|b\rangle$,

$$\left|\,|a\rangle + |b\rangle\,\right|^2 + \left|\,|a\rangle - |b\rangle\,\right|^2 = 2(\langle a|a\rangle + \langle b|b\rangle)$$