Lecture 4: Worked Examples

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Problem 1

Show that the composition of two unitary transformations $\boldsymbol{U}$ and $\boldsymbol{V}$, is also unitary.

$$\text{i.e, } (\boldsymbol{UV})^\dagger = (\boldsymbol{UV})^{-1}$$

Solution: We start with,

$$\begin{aligned}
(\boldsymbol{UV})^\dagger &= \boldsymbol{V}^\dagger \boldsymbol{U}^\dagger \quad \ldots\text{shown earlier} \\
&= \boldsymbol{V}^{-1}\boldsymbol{U}^{-1}, \quad \ldots \boldsymbol{U} \text{ and } \boldsymbol{V} \text{ are unitary} \\
&= (\boldsymbol{UV})^{-1}
\end{aligned}$$

Therefore, $\boldsymbol{UV}$ is also unitary!

Problem 2

Consider an operator $\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ in some basis $\mathbb{B} = \Bigg\{ \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{|e_1\rangle}, \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{|e_2\rangle} \Bigg\}$

Give representation of $\boldsymbol{A}$ in some $\mathbb{B}' = \Bigg\{ \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{|e_1'\rangle}, \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{|e_2'\rangle} \Bigg\}$.

Solution: To get $\boldsymbol{A}' = \boldsymbol{U}^\dagger \boldsymbol{A} \boldsymbol{U}$

$$\begin{aligned} \text{we require } \boldsymbol{U} &= \begin{bmatrix} \langle e_1|e_1'\rangle & \langle e_1|e_2'\rangle \\ \langle e_2|e_1'\rangle & \langle e_2|e_2'\rangle \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ \text{and } \boldsymbol{U}^\dagger &= \begin{bmatrix} \langle e_1'|e_1\rangle & \langle e_1'|e_2\rangle \\ \langle e_2'|e_1\rangle & \langle e_2'|e_2\rangle \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \end{aligned}$$

$$\text{yielding, } \boldsymbol{A}' = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Problem 3

If $\boldsymbol{A}$ is unitary, show that

a. $\boldsymbol{A}^{-1}$ is unitary

b. $\boldsymbol{A}^{T}$ is unitary

Solution:

To prove $\boldsymbol{A}^{-1}$ is unitary, we start with

$$\begin{aligned} \boldsymbol{A}^{-1}(\boldsymbol{A}^{-1})^{\dagger} &= \boldsymbol{A}^{\dagger}(\boldsymbol{A}^{\dagger})^{\dagger} \quad \ldots \boldsymbol{A} \text{ is unitary} \\ &= (\boldsymbol{A}^{\dagger}\boldsymbol{A})^{\dagger} \\ &= \boldsymbol{I} \end{aligned}$$

To prove A^{T} is also unitary, we write

$$\begin{aligned} \boldsymbol{A}^{T}(\boldsymbol{A}^{T})^{\dagger} &= \boldsymbol{A}^{T}(\boldsymbol{A}^{\dagger})^{T} \\ &= (\boldsymbol{A}^{\dagger}\boldsymbol{A})^{T} \\ &= \boldsymbol{I} \end{aligned}$$

Problem 4

Consider a basis $\mathbb{B} = \{|e_1\rangle, |e_2\rangle, \dots\}$. Show that the matrix elements of an operator $\boldsymbol{A}$ are

$$\boldsymbol{A}_{ij} = \langle e_i|\boldsymbol{A}|e_j\rangle$$

Solution: Let's consider an operation

$$\begin{aligned} |a\rangle &= \boldsymbol{A}\,|b\rangle \\ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \end{bmatrix} &= \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} & \dots \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \end{bmatrix} \\ \text{yielding } a_i &= \sum_j \boldsymbol{A}_{ij} b_j \end{aligned}$$

But we note from above operation,

$$a_i = \langle e_i|a\rangle = \langle e_i|\boldsymbol{A}|b\rangle = \sum_j \langle e_i|\boldsymbol{A}|e_j\rangle\, b_j$$

Comparing these two, we get $\boldsymbol{A}_{ij} = \langle e_i|\boldsymbol{A}|e_j\rangle$

Problem 5

Prove that $\boldsymbol{A}^{\dagger} = (\boldsymbol{A}^{*})^{T}$

Solution

The matrix elements of $\boldsymbol{A}$ in some basis $\mathbb{B} = \{|e_1\rangle, |e_2\rangle \dots\}$ are

$$\boldsymbol{A}_{ij} = \boldsymbol{A}^{T}_{ji} = \langle e_i|\boldsymbol{A}|e_j\rangle = \langle e_j|\boldsymbol{A}^{\dagger}|e_i\rangle^{*} = (\boldsymbol{A}^{\dagger}_{ji})^{*}$$

Conjugating both sides,

$$(\boldsymbol{A}^{T*}_{ji}) = \boldsymbol{A}^{\dagger}_{ji}$$

Implying that

$$\boldsymbol{A}^{T*} = \boldsymbol{A}^{\dagger}$$