Lecture 5: Basis Transformations - II

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Generalizing to Arbitrary Bases

Consider

$$\mathbb{B} = \{|v_1\rangle, |v_2\rangle, \ldots |v_N\rangle\} \xrightarrow{\text{transformed to}} \mathbb{B}' = \{|v_1'\rangle, |v_2'\rangle, \ldots |v_N'\rangle\}$$

with an arbitrary ket

$$|a\rangle = \underbrace{\sum_i a_i |v_i\rangle}_{\mathbb{B}} = \underbrace{\sum_j a_j' |v_j'\rangle}_{\mathbb{B}'}$$

How do the components of $|a\rangle$ transform?

$$\begin{bmatrix} a_1' \\ a_2' \\ \vdots \end{bmatrix} \xleftarrow{\ ?\ } \begin{bmatrix} a_1 \\ a_2 \\ \vdots \end{bmatrix}$$

To answer this, we must find the **similarity** transformation!

Transformation of Base Kets

Since both bases are complete

$$|v_i'\rangle = \sum_j \boldsymbol{S}_{ji} \, |v_j\rangle$$

$$|v_j\rangle = \sum_k \boldsymbol{S}^{-1}_{kj} \, |v_k'\rangle$$

Proof is very easy!

Let's take $\quad |v_j\rangle = \sum_k \boldsymbol{P}_{kj} \, |v_k'\rangle \quad \ldots$ assuming $\boldsymbol{P} \neq \boldsymbol{S}^{-1}$

then $\quad |v_i'\rangle = \sum_j \boldsymbol{S}_{ji} \sum_k \boldsymbol{P}_{kj} \, |v_k'\rangle = \sum_k \underbrace{\left(\sum_j \boldsymbol{P}_{kj} \boldsymbol{S}_{ji} \right)}_{(\boldsymbol{PS})_{ki}} |v_k'\rangle$

only valid if, $\quad \boldsymbol{PS} = \boldsymbol{I} \quad \ldots$ base kets are linearly independent

yielding, $\quad \boldsymbol{P} = \boldsymbol{S}^{-1} \quad \ldots$ contradicts our asumption

Transformation of an Arbitrary Ket

An arbitrary ket

$$|a\rangle = \sum_i a_i \, |v_i\rangle = \sum_i a_i \underbrace{\left(\sum_j \boldsymbol{S}_{ji}^{-1} \, |v_j'\rangle \right)}_{|v_i\rangle} = \sum_j \underbrace{\left(\sum_i \boldsymbol{S}_{ji}^{-1} a_i \right)}_{a_j'} |v_j'\rangle$$

yielding a transformation rule,

$$\underbrace{\begin{bmatrix} a_1' \\ a_2' \\ \vdots \end{bmatrix}}_{\text{new}} = \underbrace{\begin{bmatrix} S_{11}^{-1} & S_{12}^{-1} & \dots \\ S_{12}^{-1} & S_{22}^{-1} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}}_{\boldsymbol{S}^{-1}} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \end{bmatrix}}_{\text{old}}$$

Transformation of Operators

Imagine the operation

$$|a\rangle = \boldsymbol{X}\,|b\rangle$$

Matrix representation in $\mathbb{B}$

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{X}_{11} & \boldsymbol{X}_{12} & \dots \\ \boldsymbol{X}_{21} & \boldsymbol{X}_{22} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}}_{\text{known}} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \end{bmatrix}$$

Matrix representation in $\mathbb{B}'$

$$\begin{bmatrix} a'_1 \\ a'_2 \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{X}'_{11} & \boldsymbol{X}'_{12} & \dots \\ \boldsymbol{X}'_{21} & \boldsymbol{X}'_{22} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}}_{\textbf{unknown}} \begin{bmatrix} b'_1 \\ b'_2 \\ \vdots \end{bmatrix}$$

Operator Elements in New Basis

Lets recall the operation in $\mathbb{B}'$

$$\begin{bmatrix} a_1' \\ a_2' \\ \vdots \end{bmatrix} = \boldsymbol{X}' \begin{bmatrix} b_1' \\ b_2' \\ \vdots \end{bmatrix} = \boldsymbol{S}^{-1} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \end{bmatrix} = \boldsymbol{X}'\boldsymbol{S}^{-1} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \end{bmatrix}$$

Applying $\boldsymbol{S}$ (from left) to the last two,

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \end{bmatrix} = \boldsymbol{S}\boldsymbol{X}'\boldsymbol{S}^{-1} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \end{bmatrix} = \boldsymbol{X} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \end{bmatrix}$$

giving us,

$$\begin{aligned} \boldsymbol{S}\boldsymbol{X}'\boldsymbol{S}^{-1} &= \boldsymbol{X} \\ \boldsymbol{X}' &= \boldsymbol{S}^{-1}\boldsymbol{X}\boldsymbol{S} \end{aligned}$$

Application in a Problem

Consider two bases

$$\mathbb{B} = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\} \quad \text{and} \quad \mathbb{B}' = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}$$

Some linear operator $\boldsymbol{X}$ has the following representation in $\mathbb{B}$

$$\boldsymbol{X} = \begin{bmatrix}1 & 2 & -1\\ 0 & -1 & 0\\ 1 & 0 & 7\end{bmatrix}.$$

What is the representation of $\boldsymbol{X}$ in $\mathbb{B}'$?
To answer this, we need to construct $\boldsymbol{S}$ first!

Similarity Operator $\boldsymbol{S}$

Recalling that

$$\mathbb{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \mathbb{B}' = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

and

$$|v_i'\rangle = \sum_j \boldsymbol{S}_{ji} \, |v_j\rangle$$

we get

$$\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{|v_1'\rangle} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{|v_1\rangle}, \quad \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{|v_2'\rangle} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{|v_1\rangle} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{|v_2\rangle}, \quad \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{|v_3'\rangle} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{|v_1\rangle} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{|v_2\rangle} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{|v_3\rangle}$$

$$\text{yielding, } \boldsymbol{S} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \boldsymbol{S}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Transformation of Operator $\boldsymbol{X}$

Armed with $\boldsymbol{S}$ and $\boldsymbol{S}^{-1}$, we compute

$$\begin{aligned}\boldsymbol{X}' = \boldsymbol{S}^{-1}\boldsymbol{X}\boldsymbol{S} &= \begin{bmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & -1\\ 0 & -1 & 0\\ 1 & 0 & 7\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}\\ &= \begin{bmatrix}1 & 4 & 3\\ -1 & -2 & -9\\ 1 & 1 & 8\end{bmatrix}\end{aligned}$$

This is the matrix representation of $\boldsymbol{X}$ in $\mathbb{B}'$