Lecture 7: Hermitian Operators

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Hermitian Operators

- Represent physical observables such as spin, energy ...
- Mathematically equal to their Hermitian adjoint, $\boldsymbol{A} = \boldsymbol{A}^\dagger$

Theorem

Their eigenvalues are real and the eigenkets belonging to distinct eigenvalues are orthogonal

Proof: Pick two eigenkets $|v_i\rangle$ and $|v_j\rangle$ with eigenvalues $\lambda_i \neq \lambda_j$

$$
\begin{aligned}
\langle v_i|\boldsymbol{A}|v_j\rangle &= \lambda_j \langle v_i|v_j\rangle && \ldots \boldsymbol{A}|v_j\rangle = \lambda_j |v_j\rangle \\
&= \langle v_j|\boldsymbol{A}|v_i\rangle^* && \ldots \boldsymbol{A}^\dagger = \boldsymbol{A} \\
&= \lambda_i^* \langle v_i|v_j\rangle && \ldots \langle v_j|v_i\rangle^* = \langle v_i|v_j\rangle
\end{aligned}
$$

Subtracting last from first, $(\lambda_j - \lambda_i^*)\langle v_i|v_j\rangle = 0$

For $j = i$, we get $\lambda_j = \lambda_j^*$ "Eigenvalues are **real**"

For $j \neq i$, we get $\langle v_i|v_j\rangle = 0$ "Eigenkets are **orthogonal**"

Eigenbasis of Hermitian $\boldsymbol{A}$

The eigenkets can be normalized to set up an orthonormal basis

$$\mathbb{V} = \{|v_1\rangle, |v_2\rangle, \ldots |v_N\rangle\} \quad \text{with} \;\; \langle v_i|v_j\rangle = \delta_{ij}$$

It is easy to see that $\boldsymbol{A}$ is diagonal in this **eigenbasis**, i.e

$$\langle v_k|\boldsymbol{A}|v_j\rangle = \lambda_j \delta_{ij} = (\boldsymbol{A}_D)_{ij}$$

Representation of $\boldsymbol{A}$ in $\mathbb{V}$

Consider a typical vector $|a\rangle = \sum_l a_l |v_l\rangle$

Using closure of orthonormal $\mathbb{V}$, we write $\boldsymbol{A} = \sum_{ij} |v_i\rangle \langle v_i|\boldsymbol{A}|v_j\rangle \langle v_j|$

The k^{th} component of $\boldsymbol{A}|a\rangle$,

$$\langle v_k|\boldsymbol{A}|a\rangle = \sum_j \underbrace{\langle v_k|\boldsymbol{A}|v_j\rangle}_{\boldsymbol{A}_{kj}} a_j$$

Diagonalization of Hermitian $\boldsymbol{A}$

- Given $\boldsymbol{A}$ in some orthonormal basis $\mathbb{B} = \{|e_1\rangle, |e_2\rangle, \ldots\}$
- Construct the eigenbasis of $\boldsymbol{A}$, say $\mathbb{V} = \{|v_1\rangle, |v_2\rangle, \ldots\}$
- Transforming $\mathbb{B}$ to $\mathbb{V}$ will diagonalize $\boldsymbol{A}$
- Both $\mathbb{B}$ and $\mathbb{V}$ are orthonormal, transformation will be unitary!

Procedure

$$
\begin{aligned}
(\boldsymbol{A}_D)_{ij} &= \langle v_i|\boldsymbol{A}|v_j\rangle \quad \ldots \text{ matrix elements in } \mathbb{V} \\
&= \sum_{kl} \langle v_i|e_k\rangle \langle e_k|\boldsymbol{A}|e_l\rangle \langle e_l|v_j\rangle \quad \ldots \text{ closure of } \mathbb{B} \\
&= \sum_{kl} \boldsymbol{U}^\dagger_{ik} \boldsymbol{A}_{kl} \boldsymbol{U}_{lj} \\
&= (\boldsymbol{U}^\dagger \boldsymbol{A} \boldsymbol{U})_{ij}
\end{aligned}
$$

Yielding the diagonalization, $\boxed{\boldsymbol{A}_\text{D} = \boldsymbol{U}^\dagger \boldsymbol{A} \boldsymbol{U}}$

Problems for Hermitian $\boldsymbol{A}$

Pb. Show that,

$$\begin{aligned} \text{Tr}\,(\boldsymbol{A}) &= \sum_i \lambda_i \quad \ldots \text{ sum of eigenvalues} \\ \det\,(\boldsymbol{A}) &= \prod_i \lambda_i \quad \ldots \text{ product of eigenvalues} \end{aligned}$$

Proof

Clearly,

$$\sum_i \lambda_i = \text{Tr}\,(\boldsymbol{A}_D) = \text{Tr}\,(\boldsymbol{U}^\dagger \boldsymbol{A}\boldsymbol{U}) = \text{Tr}\,(\boldsymbol{A}\boldsymbol{U}\boldsymbol{U}^\dagger) = \text{Tr}\,(\boldsymbol{A})$$

Similarly,

$$\prod_i \lambda_i = \det\,(\boldsymbol{A}_D) = \det\,(\boldsymbol{U}^\dagger \boldsymbol{A}\boldsymbol{U}) = \det\,(\boldsymbol{U}^\dagger \boldsymbol{U}) \cdot \det\,(\boldsymbol{A}) = \det\,(\boldsymbol{A})$$

Functions of a Hermitian $\boldsymbol{A}$

Pb. Consider a function

$$e^{\boldsymbol{A}} \equiv 1 + \boldsymbol{A} + \frac{\boldsymbol{A}^2}{2!} + \frac{\boldsymbol{A}^3}{3!} \dots$$

Such functions are used as time evolution operators in quantum mechanics. Show that the trace and determinant are respectively,

$$\text{Tr}\ (e^{\boldsymbol{A}}) = \sum_{i=1}^{N} e^{\lambda_i}$$

$$\det\ (e^{\boldsymbol{A}}) = e^{\sum_{i=1}^{N} \lambda_i}$$

where λ_i are the eigenvalues of $\boldsymbol{A}$

Functions of a Hermitian $\boldsymbol{A}$

$$
\begin{aligned}
e^{\boldsymbol{A}} = e^{\boldsymbol{U}\boldsymbol{A}_{\mathrm{D}}\boldsymbol{U}^\dagger} &= 1 + \boldsymbol{U}\boldsymbol{A}_{\mathrm{D}}\boldsymbol{U}^\dagger + \frac{1}{2!}(\boldsymbol{U}\boldsymbol{A}_{\mathrm{D}}\boldsymbol{U}^\dagger)^2 + \frac{1}{3!}(\boldsymbol{U}\boldsymbol{A}_{\mathrm{D}}\boldsymbol{U}^\dagger)^3 \ldots \\
&= \boldsymbol{U}\left(1 + \boldsymbol{A}_{\mathrm{D}} + \frac{1}{2!}\boldsymbol{A}_{\mathrm{D}}^2 + \frac{1}{3!}\boldsymbol{A}_{\mathrm{D}}^3 \ldots\right)\boldsymbol{U}^\dagger \\
&= \boldsymbol{U}\begin{bmatrix} \sum_{k=0}^{\infty} \frac{\lambda_1^k}{k!} & 0 & \ldots \\ 0 & \sum_{k=0}^{\infty} \frac{\lambda_2^k}{k!} & \ldots \\ \vdots & \vdots & \ddots \end{bmatrix}\boldsymbol{U}^\dagger \\
&= \boldsymbol{U}\begin{bmatrix} e^{\lambda_1} & 0 & \ldots \\ 0 & e^{\lambda_2} & \ldots \\ \vdots & \vdots & \ddots \end{bmatrix}\boldsymbol{U}^\dagger
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathrm{Tr}\,(e^{\boldsymbol{A}}) &= \sum_{i=1}^{N} e^{\lambda_i} \quad \ldots \text{Tr is cyclic} \\
\det\,(e^{\boldsymbol{A}}) &= e^{\sum_{i=1}^{N}\lambda_i} \quad \ldots \det\,(\boldsymbol{U}\boldsymbol{U}^\dagger) = 1
\end{aligned}
$$