Lecture 9: Worked Examples

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Problem: Simultaneous Diagonalization

Q. Take two operators in the standard basis

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Find the transform that will simultaneously diagonalize **A** and **B**

Solution

Lets find out if they share an eigenbasis. To this end, we compute

$$[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA} = 0$$

They do share an eigenbasis, say $\mathbb{V} = \{|v_1\rangle, |v_2\rangle, |v_3\rangle\}$

Transforming to $\mathbb{V}$ will simultaneously diagonalize **A** and **B**

Problem: Simultaneous Diagonalization

The characteristic equation for **A**,

$$0 = \det\left(\mathbf{A} - a\mathbf{I}\right) = a^2(2-a), \quad \text{yields } a = 0, 0, 2$$

For the non-repeated eigenvalue, $a = 2$:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix}}_{\mathbf{A}-2\mathbf{I}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{|a\rangle} = \begin{bmatrix} -x+z \\ -2y \\ x-z \end{bmatrix}, \quad \text{we get } x = z, y = 0$$

We pick $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \dfrac{1}{\sqrt{2}} = |v_1\rangle$, as a shared eigenket

Problem: Simultaneous Diagonalization

For the repeated eigenvalue, $a = 0$:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{\mathbf{A}-0\mathbf{I}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{|a\rangle} = \begin{bmatrix} x+z \\ 0 \\ x+z \end{bmatrix}, \quad \text{we get } x = -z, y = \text{arbitrary}$$

Picking two orthonormal kets, $|0_1\rangle = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \frac{1}{\sqrt{2}}$, and $|0_2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Being degenerate, $|0_1\rangle$ & $|0_2\rangle$ are not necessarily eigenkets of **B**
But linear combinations of $|0_1\rangle$ & $|0_2\rangle$ can become eigenkets of **B**!

The characteristic equation for **B**,

$$0 = \det\left(\mathbf{B} - b\mathbf{I}\right) = (b^2 - 2b - 3)(2 - b), \quad \text{yields } b = -1, 2, 3$$

Problem: Simultaneous Diagonalization

Pick a linear combination,

$$|v\rangle = \alpha |0_1\rangle + \beta |0_2\rangle = \frac{\alpha}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha/\sqrt{2} \\ \beta \\ -\alpha/\sqrt{2} \end{bmatrix}$$

To extract α and β, we use the eigenvalue equation of $\mathbf{B}$

$$\mathbf{B}|v\rangle = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} \alpha/\sqrt{2} \\ \beta \\ -\alpha/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \alpha/\sqrt{2} + \beta \\ \sqrt{2}\alpha \\ -\alpha/\sqrt{2} - \beta \end{bmatrix} = b \begin{bmatrix} \alpha/\sqrt{2} \\ \beta \\ -\alpha/\sqrt{2} \end{bmatrix}$$

For $b = -1 : \beta = -\sqrt{2}\alpha$	For $b = 2 : \beta = \alpha/\sqrt{2}$
$\|v_2\rangle = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \frac{1}{\sqrt{6}}$	$\|v_3\rangle = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \frac{1}{\sqrt{3}}$

Problem: Simultaneous Diagonalization

The unitary transform simultaneously diagonalizing **A** and **B**

$$\mathbf{U} = \begin{bmatrix} \vdots & \vdots & \vdots \\ |v_1\rangle & |v_2\rangle & |v_3\rangle \\ \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} \\ 0 & -\sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{3}} \end{bmatrix}$$

One can verify (left as exercise),

$$\mathbf{U}^\dagger \mathbf{A} \mathbf{U} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{A}_\mathrm{D}$$

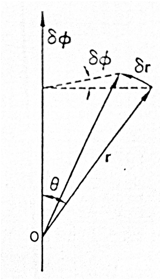
$$\mathbf{U}^\dagger \mathbf{B} \mathbf{U} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \mathbf{B}_\mathrm{D}$$

Solved

Hermitian Operators in Classical Mechanics

The angular momentum (about origin) of a point mass m

$$
\begin{aligned}
\vec{L} &= m\vec{r} \times \vec{v} \\
&= m\vec{r} \times (\vec{\omega} \times \vec{r}) \quad \ldots \vec{\omega} = \frac{d\phi}{dt}\vec{z} \\
&= m(r^2\vec{\omega} - (\vec{r} \cdot \vec{\omega})\vec{r}) \quad \ldots \vec{L} \nparallel \vec{\omega}
\end{aligned}
$$



If, $\vec{r} \perp \vec{\omega}$ (planar motion), then

$$\vec{L} = mr^2\vec{\omega} = \mathcal{I}\vec{\omega} \quad \ldots \vec{L} \parallel \vec{\omega}$$

where the scalar, $\mathcal{I} = mr^2$ is called the moment of inertia

Moment of Inertia Matrix

Recalling the general case,

$$\vec{L} = m(r^2\vec{\omega} - (\vec{r}\cdot\vec{\omega})\vec{r})$$

With $\vec{L} = \begin{bmatrix} L_1 & L_2 & L_3 \end{bmatrix}^{\mathsf{T}}, \vec{r} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^{\mathsf{T}}$ & $\vec{\omega} = \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 \end{bmatrix}^{\mathsf{T}}$,

$$\begin{aligned} L_1 &= m(x_2^2 + x_3^2)\omega_1 - mx_1x_2\omega_2 - mx_1x_3\omega_3 \\ L_2 &= -mx_2x_1\omega_1 + m(x_3^2 + x_1^2)\omega_2 - mx_2x_3\omega_3 \\ L_3 &= -mx_3x_1\omega_1 - mx_3x_2\omega_2 + m(x_2^2 + x_1^2)\omega_3 \end{aligned}$$

In matrix form,

$$\underbrace{\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}}_{|L\rangle} = \underbrace{\begin{bmatrix} m(x_2^2 + x_3^2) & -mx_1x_2 & -mx_1x_3 \\ -mx_2x_1 & m(x_3^2 + x_1^2) & -mx_2x_3 \\ -mx_3x_1 & -mx_3x_2 & m(x_2^2 + x_1^2) \end{bmatrix}}_{\mathcal{I}} \underbrace{\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}}_{|\omega\rangle}$$

Problem: Rigid Body Motion

Q. For a rigid body, the elements of inertia matrix are calculated under a volume integral. The resulting inertia matrix can then be visualized as a linear operator, say,

$$\mathcal{I} = \begin{bmatrix} \frac{5}{2} & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{2}} & \frac{7}{3} & \sqrt{\frac{1}{18}} \\ \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{18}} & \frac{13}{6} \end{bmatrix} \quad \ldots \text{in some } \mathbb{B} = \{|e_1\rangle, |e_2\rangle, |e_3\rangle\}$$

Find the principal axes and the principal moments of inertia of this rigid body.

Solution

Note that $\mathcal{I}$ is diagonal in its eigenbasis $\mathbb{V} = \underbrace{\{|v_1\rangle, |v_2\rangle, |v_3\rangle\}}_{\text{principal axes}}$

Eigenvalues λ_1, λ_2 and λ_3 will be the principal moments of inertia

Problem: Rigid Body Motion

From the characteristic equation

$$0 = \det(\mathcal{I} - \lambda \mathbf{I}) = \lambda^3 - 7\lambda^2 + 14\lambda - 8 = (\lambda - 1)(\lambda - 2)(\lambda - 4)$$

$\lambda = 1$

$(\mathcal{I} - \mathbf{I}) |v\rangle = 0$ yields,

$$\begin{aligned} \frac{3}{2}x + \sqrt{\frac{3}{2}}y + \sqrt{\frac{3}{4}}z &= 0 \\ \sqrt{\frac{3}{2}}x + \frac{4}{3}y + \sqrt{\frac{1}{18}}z &= 0 \\ \sqrt{\frac{3}{4}}x + \sqrt{\frac{1}{18}}y + \frac{7}{6}z &= 0 \end{aligned}$$

for the family $x = -\sqrt{3}a, y = \sqrt{2}a, z = a \quad \ldots a \in \mathbb{R}$

picking a normalized $\boxed{|v_1\rangle = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}^{\mathsf{T}}}$

Problem: Rigid Body Motion

$\lambda = 2$

$(\mathcal{I} - 2\mathbf{I})\,|v\rangle = 0$ yields,

$$\begin{aligned} \frac{1}{2}x + \sqrt{\frac{3}{2}}y + \sqrt{\frac{3}{4}}z &= 0 \\ \sqrt{\frac{3}{2}}x + \frac{1}{3}y + \sqrt{\frac{1}{18}}z &= 0 \\ \sqrt{\frac{3}{4}}x + \sqrt{\frac{1}{18}}y + \frac{1}{6}z &= 0 \end{aligned}$$

for the family $\quad x = 0, y = -a/\sqrt{2}, z = a \quad \ldots a \in \mathbb{R}$

picking a normalized $\boxed{|v_2\rangle = \begin{bmatrix} 0 & 1/\sqrt{3} & -\sqrt{2/3} \end{bmatrix}^{\mathbf{T}}}$

Problem: Rigid Body Motion

$$\lambda = 4$$

$(\mathcal{I} - 4\mathbf{I})\,|v\rangle = 0$ yields,

$$\begin{aligned}
-\frac{3}{2}x + \sqrt{\frac{3}{2}}y + \sqrt{\frac{3}{4}}z &= 0 \\
\sqrt{\frac{3}{2}}x - \frac{5}{3}y + \sqrt{\frac{1}{18}}z &= 0 \\
\sqrt{\frac{3}{4}}x + \sqrt{\frac{1}{18}}y - \frac{11}{6}z &= 0
\end{aligned}$$

for the family $x = \sqrt{3}a, y = \sqrt{2}a, z = a \quad \ldots a \in \mathbb{R}$

picking a normalized $\boxed{|v_3\rangle = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}^{\mathbf{T}}}$

Problem: Rigid Body Motion

We summarize below

- Eigenkets are orthonormal $\langle v_i | v_j \rangle = \delta_{ij}$, since $\mathcal{I}$ is Hermitian
- In the eigenbasis $\mathbb{V} = \{|v_1\rangle, |v_2\rangle, |v_3\rangle\}$, operator $\mathcal{I}$ is diagonal

$$\underbrace{\begin{bmatrix} L_1' \\ L_2' \\ L_3' \end{bmatrix}}_{|L\rangle} = \underbrace{\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}}_{\mathcal{I}} \underbrace{\begin{bmatrix} \omega_1' \\ \omega_2' \\ \omega_3' \end{bmatrix}}_{|\omega\rangle}$$

- Eigenkets $|v_1\rangle, |v_2\rangle, |v_3\rangle$ are therefore the principal axes
- Eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are principal moments of inertia
- For eg., if the body rotates only about $|v_3\rangle$, then

$$L_1' = 0, \quad L_2' = 0, \quad L_3' = \lambda_3 \omega_3'$$