Lecture 12: Worked Examples

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Useful Results

For differentiable $f(z)$ and $g(z)$, the limit formula leads us to

$$(f(z) + g(z))' = f'(z) + g'(z)$$

$$(f(z)\,g(z))' = f'(z)\,g(z) + f(z)\,g'(z)$$

$$\left(\frac{f(z)}{g(z)}\right)' = \frac{g(z)\,f'(z) - f(z)\,g'(z)}{g(z)^2}, \quad g(z) \neq 0$$

$$[f(g(z))]' = f'(g(z))\,g'(z)$$

To differentiate polynomials, we just need

$$(z^n)' = nz^{n-1}, \quad n > 0$$

Analytic or not?

$$\boxed{f(z) = \sin(z)}$$

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{\sin(z + \Delta z) - \sin(z)}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{\sin(z)\cos(\Delta z) + \sin(\Delta z)\cos(z) - \sin(z)}{\Delta z} \\ &= \cos(z) \end{aligned}$$

exists only for finite z since $f(z)$ itself blows up as $y \to \infty$

$$\begin{aligned} \lim_{y \to \infty} |\sin(z)| &= \lim_{y \to \infty} |\sin(x)\cos(iy) + \sin(iy)\cos(x)| \\ &= \lim_{y \to \infty} |\sin(x)\cosh(y) + i\sinh(y)\cos(x)| \\ &\to \infty \end{aligned}$$

$\sin(z)$ is therefore analytic in the finite z-plane (entire)

Analytic or not?

$$\boxed{f(z) = \cos(z)}$$

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{\cos(z + \Delta z) - \cos(z)}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{\cos(z)\cos(\Delta z) - \sin(\Delta z)\sin(z) - \cos(z)}{\Delta z} \\ &= -\sin(z) \end{aligned}$$

exists only for finite z since $f(z)$ itself blows up as $y \to \infty$

$$\begin{aligned} \lim_{y \to \infty} |\cos(z)| &= \lim_{y \to \infty} |\cos(x)\cos(iy) - \sin(iy)\sin(x)| \\ &= \lim_{y \to \infty} |\cos(x)\cosh(y) - i\sinh(y)\sin(x)| \\ &\to \infty \end{aligned}$$

$\cos(z)$ is therefore analytic in the finite z-plane (entire)

Analytic or not?

$$\boxed{f(z) = \tan(z)}$$

$$f'(z) = \left(\frac{\sin(z)}{\cos(z)}\right)' = \sec^2(z)$$

$\tan(z)$ is analytic everywhere except $z \neq (2n+1)\dfrac{\pi}{2}, n \in \mathbb{Z}$

$$\boxed{f(z) = e^{\sin(z)}}$$

$$f'(z) = e^{\sin(z)} \cos(z)$$

$e^{\sin(z)}$ is analytic everywhere in the finite z-plane (entire)

$$\boxed{f(z) = e^{z^*}}$$

$$f'(z) = e^{z^*}\ z^{*\prime}$$

e^{z^*} is analytic nowhere

Analytic or not?

$$\boxed{f(z) = \frac{z}{z^n+1}, \text{integer } n > 0}$$

$$f'(z) = \frac{z^n(1-n)+1}{(z^n+1)^2}$$

$f(z)$ analytic everywhere except at the n roots of the polynomial $z^n + 1 = 0$, namely at $z = e^{i(\pi+2m\pi)/n}$ with $m = 0, 1, \ldots n-1$.

$$\boxed{f(z) = \text{Re } (z) = x}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{x + \Delta x - x}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta x}{\Delta z} = \begin{cases} 1, & \Delta z = \Delta x \\ 0, & \Delta z = i\Delta y \end{cases}$$

Re (z) is not analytic anywhere

Analytic or not?

$$\boxed{f(z) = \sqrt{z}}$$

Rewriting as

$$f(z) = \sqrt{z} = \sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$

and using polar CR

$$\frac{\partial u}{\partial r} = \frac{1}{2\sqrt{r}}\cos\frac{\theta}{2} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial r} = \frac{1}{2\sqrt{r}}\sin\frac{\theta}{2} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

$f(z)$ is analytic in the region $r > 0, \alpha < \theta < \alpha + 2\pi$ with $\alpha \in \mathbb{R}$

$$\text{with } f'(z) = e^{-i\theta}\frac{\partial f}{\partial r} = \frac{1}{2f(z)}$$

Recovering f from u

Find the analytic $f \equiv f(z)$, with real part

$$\boxed{u(x,y) = 3x^2y - y^3}$$

$$\frac{\partial u}{\partial x} = 6xy = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = 3(x^2 - y^2) = -\frac{\partial v}{\partial x}$$

Integrating,

$$v(x,y) = 3xy^2 + c_1(x) = 3xy^2 - x^3 + c_2(y)$$

Implying,

$$c_1(x) = -x^3 + c \quad \text{and} \quad c_2(y) = c$$

Yielding,

$$\begin{aligned} f(z) = u + i\ v &= 3x^2y - y^3 + i(3xy^2 - x^3 + c) \\ &= -i(x^3 + 3x(iy)^2 - c + (iy)^3 + 3x^2(iy)) \\ &= -i[z^3 - c] \end{aligned}$$

Recovering f from u

Find the analytic $f \equiv f(z)$, with real part

$$\boxed{u(x,y) = \cos(x)\cosh(y)}$$

$$\frac{\partial u}{\partial x} = -\sin(x)\cosh(y) = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = \cos x \sinh(y) = -\frac{\partial v}{\partial x}$$

Integrating,

$$v(x,y) = -\sin(x)\sinh(y) + c_1(x) = -\sin(x)\sinh(y) + c_2(y)$$

Implying,

$$c_1(x) = c_2(y) = c$$

Yielding,

$$\begin{aligned} f(z) = u + i\,v &= \cos(x)\cosh(y) - i\sin(x)\sinh(y) + c \\ &= \cos(x)\cos(iy) - \sin(x)\sin(iy) + c \\ &= \cos(z) + c \end{aligned}$$

Basic Trigonometric Formulas

sin() and cos() are the fundamental orthogonal functions, implying

$$\langle \cos(mx), \sin(nx) \rangle = \frac{1}{\pi} \int_{2\pi} \cos(mx) \sin(nx) \mathrm{d}x = 0 \quad \ldots m, n \in \mathbb{Z}$$

Using sin() and cos(), we construct other trigonometric functions

Analytic Series

For finite z,

$$\sin(z)' = \cos(z) \quad \text{and} \quad \cos(z)' = -\sin(z)$$

Therefore, a Taylor expansion about origin gives

$$\begin{aligned}
\sin(z) &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \sin^{(n)}(0) = z - \frac{z^3}{3!} + \frac{z^5}{5!} \ldots \\
i \sin(z) &= iz + \frac{(iz)^3}{3!} + \frac{(iz)^5}{5!} \ldots \\
\cos(z) &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \cos^{(n)}(0) = 1 + \frac{(iz)^2}{2!} + \frac{(iz)^4}{4!} \ldots
\end{aligned}$$

Basic Trigonometric Formulas

Adding

$$\cos(z) + i\sin(z) = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = \sum_{n=0}^{\infty} i^n \frac{z^n}{n!} = e^{iz}$$

One can get series for

$$\tan(z) = \frac{\sin(z)}{\cos(z)} = \frac{1}{i} \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}$$

One can show that for a finite z,

$$\cos^2(z) + \sin^2(z) = \cos^2(z) - (i\sin(z))^2 = e^{iz}\, e^{-iz} = 1$$

Pb. Show that $f(z) = \cos^2(z) + \sin^2(z) - 1$, is zero for finite z
Solution: Since $f(0) = 0$ and $f'(z) = 0$ for finite z, we establish

$$f(z) = 0 \quad \ldots \text{finite } z$$