Lecture 13: Harmonic Pieces of f(z)

Ashwin Joy

Department of Physics, IIT Madras, Chennai - 600036

Analytical Functions have Harmonic Pieces

$f(z) = u(x, y) + iv(x, y)$ is analytic in some $\mathcal{R}$, if and only if

$$\boxed{\nabla^2 u = 0 = \nabla^2 v}$$

"u and v are harmonic in $\mathcal{R}$"

Proof

For necessary condition, we start from Cauchy-Riemann,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

The Laplacian,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$$

As the second derivatives are continuous, they are also symmetric

Vice Versa is Also True

If $\nabla^2 u = 0$, then u is the real part of some analytic f

Proof

We construct an analytic function

$$\begin{aligned} g(z) &= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} && \ldots u \text{ is harmonic} \\ &= f'(z) && \ldots f \text{ is the anti-derivative of } g \\ &= \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} && \ldots f = U + iV \\ &= \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y} && \ldots g \text{ is unique so } f \text{ satisfies CR} \end{aligned}$$

On comparing the first and the last equation,

$$\boxed{u = U + \text{constant}}$$

Symmetry of Second Derivatives - Stokes Theorem

For a vector field $\boldsymbol{A}$ in some flat region $\mathcal{R}$ bounded by a curve $\mathcal{C}$

$$\iint_{\mathcal{R}} (\boldsymbol{\nabla} \times \boldsymbol{A}) \cdot \mathrm{d}\boldsymbol{\mathcal{S}} = \oint_{\mathcal{C}} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l}$$

Substituting $\boldsymbol{A} = (u, v)$, $\mathrm{d}\boldsymbol{\mathcal{S}} = \mathrm{d}x\, \mathrm{d}y\, \hat{\boldsymbol{z}}$ and $\mathrm{d}\boldsymbol{l} = (\mathrm{d}x, \mathrm{d}y)$, gives

$$\boxed{\iint_{\mathcal{R}} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \mathrm{d}x\, \mathrm{d}y = \oint_{\mathcal{C}} (u\, \mathrm{d}x + v\, \mathrm{d}y) \quad \text{— Green's theorem}}$$

which requires u, v and their partial derivatives to be continuous.

Finally substituting $(u, v) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = \boldsymbol{\nabla} f$, in Green's theorem

$$\iint_{\mathcal{R}} \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) \mathrm{d}x\, \mathrm{d}y = \oint_{\mathcal{C}} \left(\frac{\partial f}{\partial x}\, \mathrm{d}x + \frac{\partial f}{\partial y}\, \mathrm{d}y \right) = \oint_{\mathcal{C}} \mathrm{d}f = 0$$

Since the region $\mathcal{R}$ is arbitrary, we can demand that the integrand in the surface integral vanishes, leading us to

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}}$$

the symmetry of mixed derivatives when $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ and their partial derivatives are continuous.

Symmetry of Second Derivatives - Divergence Theorem

We can also start with the 2D version of the divergence theorem -

$$\iint_{\mathcal{R}} (\boldsymbol{\nabla} \cdot \boldsymbol{A})\, \mathrm{d}\mathcal{S} = \oint_{\mathcal{C}} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{p}$$

where $d\boldsymbol{p} = (\mathrm{d}y, -\mathrm{d}x)$ is outward normal to the line element $\mathrm{d}\boldsymbol{l} = (\mathrm{d}x, \mathrm{d}y)$ on the curve $\mathcal{C}$ bounding the region $\mathcal{R}$. Substituting $\boldsymbol{A} = (v, -u)$, we get

$$\boxed{\iint_{\mathcal{R}} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \mathrm{d}x\, \mathrm{d}y = \oint_{\mathcal{C}} (u\, \mathrm{d}x + v\, \mathrm{d}y) \quad \text{— Green's theorem}}$$

symmetry of mixed derivatives can now be proved as shown earlier.

Level Curves of u and v

Let $u(x, y) = y^2 - x^2$ be the real part of some analytic $f(z)$.
Find its harmonic conjugate and plot some level curves.

Solution

From CR,

$$\frac{\partial u}{\partial x} = -2x = \frac{\partial v}{\partial y} \implies v = -2xy + c_1(x)$$

$$\frac{\partial u}{\partial y} = 2y = -\frac{\partial v}{\partial x} \implies v = -2xy + c_2(y)$$

Implying $c_1(x) = c_2(y) = c$, thus

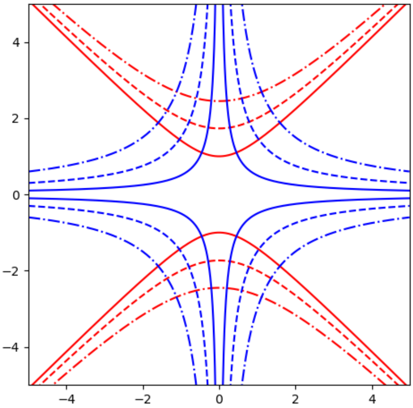
$$f(z) = y^2 - x^2 + i\left(-2xy + c\right) = -z^2 + c$$

Lets plot level curves of

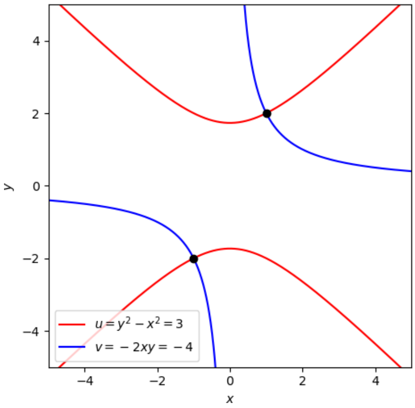
$$u = y^2 - x^2 = c_1 \quad \text{and} \quad v = -2xy = c_2$$

Level Curves of u and v

They are orthogonal to each other at every (x, y)



$u = c_1$ (red) and $v = c_2$ (blue)



Orthogonal intersection at (1,2) & (-1,-2)

Level Curves of u and v

So why are the level curves orthogonal at intersection?

Answer

Directions of steepest change in u and v are respectively along,

$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \quad \text{and} \quad \nabla v = \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right)$$

Thus we need only the angle between these two directions!
Using CR, I rewrite

$$\nabla u = \left(\frac{\partial v}{\partial y}, -\frac{\partial v}{\partial x}\right)$$

Clearly, this is orthogonal to ∇v