# Lecture 14: Ideal Fluid Flows

Ashwin Joy

Department of Physics, IIT Madras, Chennai - 600036

# Ideal Fluid Flows

- They are paradigms of Laplace equations satisfying
- Steady state: $\dfrac{\partial}{\partial t} \equiv 0$
- Zero viscosity, $\nu = 0$
- Incompressibility, $\dfrac{\mathrm{d}}{\mathrm{d}t}\rho(\boldsymbol{r}, t) = 0$
- Irrotationality, $\boldsymbol{\nabla} \times \boldsymbol{v} = 0$

# Incompressibility

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\rho(\boldsymbol{r},t) && \text{...density is always conserved} \\
&= \frac{\partial \rho}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\rho \\
&= \underbrace{-\boldsymbol{\nabla}\cdot(\rho\boldsymbol{v})}_{\text{mass continuity}} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\rho \\
&= -[\rho\boldsymbol{\nabla}\cdot\boldsymbol{v} + \cancel{\boldsymbol{v}\cdot(\boldsymbol{\nabla}\rho)}] + \cancel{(\boldsymbol{v}\cdot\boldsymbol{\nabla})\rho}
\end{aligned}
$$

yielding the continuity equation for incompressible fluids

$$
\boxed{\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0} \implies \boxed{\boldsymbol{v} = \boldsymbol{\nabla}\times\boldsymbol{\Psi}}
$$

Thus $\boldsymbol{v}$ is solenoidal — in analogy with magnetostatics. Setting the vector potential $\boldsymbol{\Psi} = \psi(x,y)\hat{\boldsymbol{z}}$, gives us the 2D velocity

$$
\boxed{\boldsymbol{v} = \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}\right)}
$$

# Irrotationality

Such flows cannot rotate a particle about its own axis, implying

$$\boxed{\nabla \times \boldsymbol{v} = 0} \implies \boxed{\boldsymbol{v} = \nabla \phi}$$

The 2D velocity in terms of the scalar potential $\phi$ is therefore

$$\boxed{\boldsymbol{v} = \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right)}$$

Next we show that $\phi$ and $\psi$ completely describe the flow!

# Ideal flow as a complex function

Did you notice that $\psi$ and $\phi$ are harmonic?

From incompressibility, $0 = \nabla \cdot \boldsymbol{v} = \nabla \cdot (\nabla \phi) = \nabla^2 \phi$

From irrotationality, $0 = \nabla \times \boldsymbol{v} = \nabla \times (\nabla \times \boldsymbol{\Psi}) = -\nabla^2 \psi \hat{z}$

Since $\boldsymbol{v}$ is unique, they also satisfy Cauchy-Riemann

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

Thus they are pieces of an analytic function

$$\boxed{\Omega(z) = \phi(x, y) + i\, \psi(x, y)}$$

whose derivative gives the local velocity

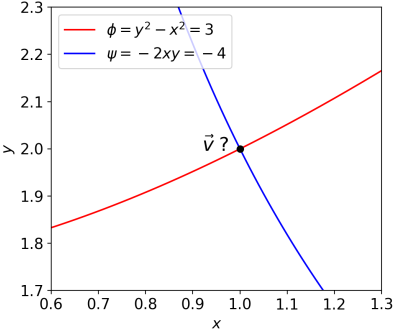
$$\boxed{\Omega'(z) = \frac{\partial \phi}{\partial x} + i\, \frac{\partial \psi}{\partial x} = v_x - i\, v_y}$$

# Level Curves of $\phi$ and $\psi$ reveal the local flow

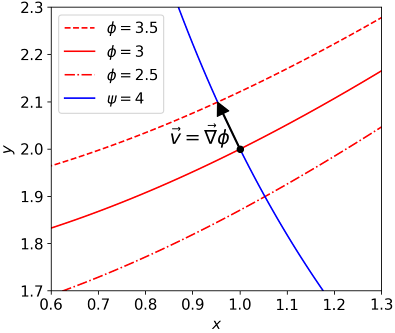
Take the ideal flow described by the function

$$\Omega(z) = -z^2 = y^2 - x^2 - i\,2xy$$

**Q:** What is the direction of flow at •?



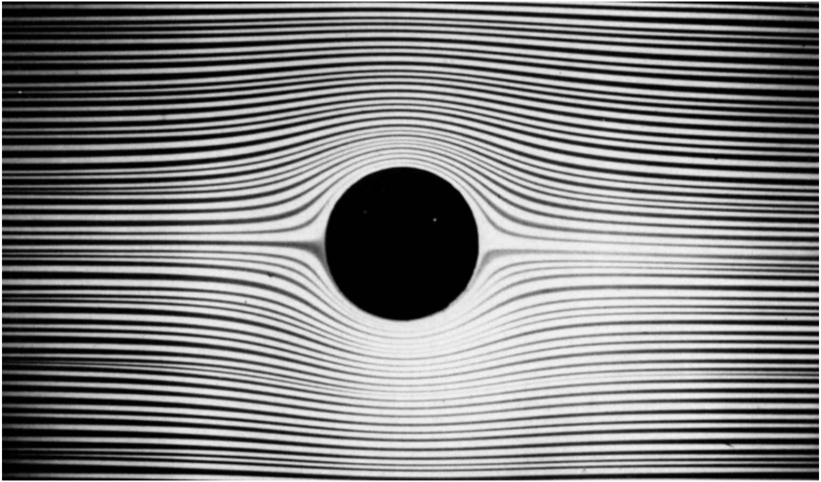

**A:** Along the streamline!




$\psi(x, y) = c_1$ ... streamlines $\parallel$ $\boldsymbol{v}$

$\phi(x, y) = c_2$ ... equipotential lines $\perp$ $\boldsymbol{v}$

# Flow Past a Circular Obstacle



Water flowing past a circular obstacle of radius $a$. Streamlines are tracked by a dye.

*Album of Fluid Motion*, Milton Van Dyke

# Experimental Observations

From the photograph, we notice the two boundary conditions

$$\text{As } r/a \gg 1, \quad \boldsymbol{v} \sim v_0 \hat{x}$$

$$\text{As } r/a \to 1, \quad \boldsymbol{v} \sim v_\theta \hat{\boldsymbol{\theta}}$$

**Big Question**

Can the flow at boundary describe the flow everywhere else?

The answer is a delightful "YES". We just need $\Omega(z)$ at the boundary. Laplace equation then demands that this solution must hold everywhere as it only admits a unique solution.

# Complex function describing the flow

Lets guess the $\Omega(z)$ at the boundaries —

$$\text{As } r/a \gg 1, \quad \Omega(z) \sim v_0 z$$

$$\text{As } r/a \to 1, \quad \Omega(z) \sim \phi(x, y)$$

where in the second limit, we have taken the streamline hugging the surface of the obstacle as $\psi = 0$, without loss of generality.

The only function that satisfies the boundary condition is

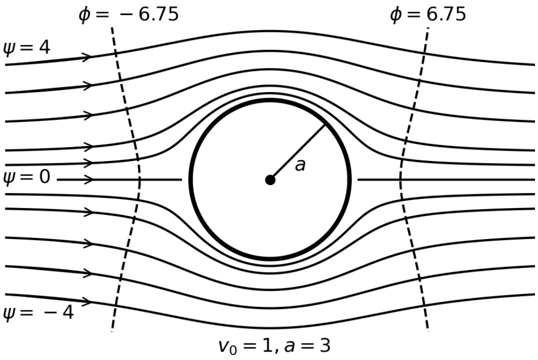
$$\boxed{\Omega(z) = v_0 z + v_0 \frac{a^2}{z}}$$

For eg., $\Omega(z) = v_0 z + v_0 \dfrac{a^3}{z^2}$ works at $r/a \gg 1$ but fails at $r/a \to 1$.

# Sketching the streamlines

Since $\Omega(z) = \phi + i\psi$, the corresponding potentials become

$$\phi = v_0\left(r + \frac{a^2}{r}\right)\cos\theta \qquad \psi = v_0\left(r - \frac{a^2}{r}\right)\sin\theta$$




recovers the experimental flow pattern!

# Velocity field

From the derivative

$$\Omega'(z) = v_0\left(1 - \frac{a^2}{z^2}\right) = v_0\left(1 - \frac{a^2 e^{-2i\theta}}{r^2}\right) = v_x - i\ v_y$$

we read the Cartesian components

$$v_x = v_0\left(1 - \frac{a^2 \cos 2\theta}{r^2}\right) \quad \text{and} \quad v_y = -v_0 \frac{a^2 \sin 2\theta}{r^2}$$

which gives $\boldsymbol{v} \sim v_0 \hat{\boldsymbol{x}}$ as $r/a \gg 1$. In polar coordinates however,

$$\boldsymbol{v} = \boldsymbol{\nabla}\phi = \frac{\partial \phi}{\partial r}\ \hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\ \hat{\boldsymbol{\theta}} = v_0\left(1 - \frac{a^2}{r^2}\right) \cos\theta\ \hat{\boldsymbol{r}} + v_0\left(1 + \frac{a^2}{r^2}\right) \sin\theta\ \hat{\boldsymbol{\theta}}$$

which gives $\boldsymbol{v} \sim 2v_0 \sin\theta\ \hat{\boldsymbol{\theta}}$ as $r/a \to 1$.

# Food for thought

**Q.** Where are the stagnation points ($\boldsymbol{v} = 0$) of the flow?
**A.** At $r = a$ and $\theta = (0, \pi)$

**Q.** Find the level curves of $\phi$ and $\psi$ at $r/a \gg 1$
**A.** In this far field region—
Streamlines $\psi \sim v_0 y =$ constant, are lines parallel to $x$-axis.
Equipotentials $\phi \sim v_0 x =$ constant, are lines parallel to $y$-axis.