Lecture 18: Series Expansions

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Taylor's Series

- If f is analytic at some z_0, it is infinitely differentiable there
- A power series then exists in the neighborhood of z_0

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad a_n = \frac{1}{n!} f^{(n)}(z_0)$$

  with a radius of convergence defined by the nearest singularity
- Converse is true, convergent power series defines an analytic f

Example

The function $f(z) = \dfrac{1}{1+z}$ exists everywhere except at $z = -1$

A Taylor expansion about origin exists

$$f(z) = \frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n \qquad \text{converges for } |z| < 1$$

Q. How do we represent $f(z)$ as a power series for $|z| > 1$?

A. By first writing $f(z) = \dfrac{1}{z(1+1/z)}$ and

Taylor expanding $\dfrac{1}{1+1/z} = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{z^n}$ about ∞ with $\dfrac{1}{|z|} < 1$

Finally, $f(z) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{1}{z^{n+1}}$ with $|z| > 1$

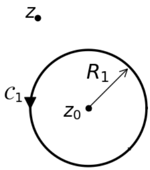
Power Series About a Singularity

- Consider z_0 as a singularity of some $f(z)$
- Clearly Taylor expansion about z_0 is not possible
- Complex algebra still allows a power series about z_0
- Expansion coefficients obtained by contour integrals about z_0
- In what follows, we show how this is done

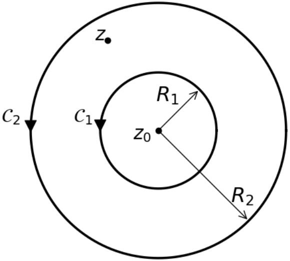
Laurent Series

Laurent Series

Laurent Series

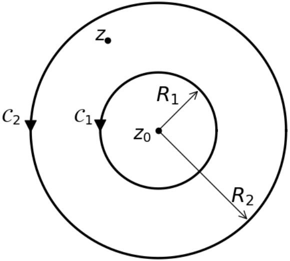


Laurent Series



Laurent Series

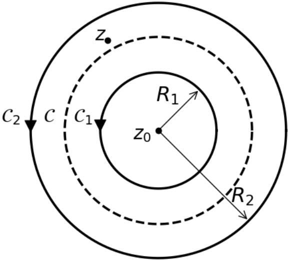
If $f(z)$ analytic in an annular region $R_1 \leq |z - z_0| \leq R_2$



$$
\begin{aligned}
f(z) &= \sum_{n=-\infty}^{\infty} C_n (z - z_0)^n \\
C_n &= \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, \mathrm{d}\zeta
\end{aligned}
$$

Laurent Series

If $f(z)$ analytic in an annular region $R_1 \leq |z - z_0| \leq R_2$

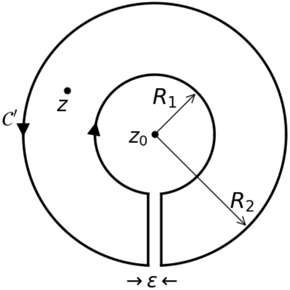


$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z - z_0)^n$$

$$C_n = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, \mathrm{d}\zeta$$

$\mathcal{C}$ encloses inner circle of radius R_1

$$f(z) = \frac{1}{2\pi i} \left[\oint_{\mathcal{C}_2} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta - \oint_{\mathcal{C}_1} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta \right]$$



Take $\epsilon \to 0$ in
Cauchy's integral formula

Laurent Series

In the first integral, we write

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0)\left(1 - \frac{z - z_0}{\zeta - z_0}\right)} = \frac{1}{(\zeta - z_0)} \sum_{j=0}^{\infty} \frac{(z - z_0)^j}{(\zeta - z_0)^j}$$

$$\text{for } \left|\frac{z - z_0}{\zeta - z_0}\right| = \left|\frac{z - z_0}{R_2}\right| < 1$$

In the second integral, we write

$$-\frac{1}{\zeta - z} = \frac{1}{(z - z_0)\left(1 - \frac{\zeta - z_0}{z - z_0}\right)} = \frac{1}{(z - z_0)} \sum_{j=0}^{\infty} \frac{(\zeta - z_0)^j}{(z - z_0)^j}$$

$$\text{for } \left|\frac{\zeta - z_0}{z - z_0}\right| = \left|\frac{R_1}{z - z_0}\right| < 1$$

Laurent Series

The expansion becomes

$$f(z) = \sum_{j=0}^{\infty} A_j (z - z_0)^j + \sum_{j=0}^{\infty} B_j (z - z_0)^{-(j+1)}$$

where

$$\begin{aligned} A_j &= \frac{1}{2\pi i} \oint_{\mathcal{C}_2} \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} \, \mathrm{d}\zeta \\ B_j &= \frac{1}{2\pi i} \oint_{\mathcal{C}_1} f(\zeta)(\zeta - z_0)^j \, \mathrm{d}\zeta \end{aligned}$$

- f is analytic in the annular region
- We can replace both $\mathcal{C}_{1,2}$ by some $\mathcal{C}$ enclosing $\mathcal{C}_1$
- Now noting that $A_{-(j+1)} = B_j$, the expansion becomes

$$\boxed{f(z) = \sum_{n=-\infty}^{\infty} C_n (z - z_0)^n} \quad \text{with} \quad \boxed{C_n = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, \mathrm{d}\zeta}$$

Remarks on Laurent Series

- Coefficient C_{-1} of the term $(z - z_0)^{-1}$ is called the residue
- Negative powers of the LS constitute the principal part of $f(z)$

Usage

- Formulae for C_n are bulky for use in practice
- Taylor expansions with appropriate substitutions is enough!
- We substantiate this with some examples next

Worked Examples

Find the Laurent expansion of

$$f(z) = \frac{1}{(z-1)(z-2)} \qquad \text{for } 1 < |z| < 2$$

Solution

We rewrite

$$f(z) = -\frac{1}{z-1} + \frac{1}{z-2} = -\frac{1}{z}\underbrace{\left(\frac{1}{1-1/z}\right)}_{f_1(z)} - \frac{1}{2}\underbrace{\left(\frac{1}{1-z/2}\right)}_{f_2(z)}$$

Since $f_1(z)$ and $f_2(z)$ have convergent power series for $1 < |z| < 2$

$$f(z) = -\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n \quad \ldots \text{LS with } C_n = \begin{cases} -1 & n < 0 \\ -1/2^{n+1} & n \geq 0 \end{cases}$$

Worked Examples

Find the first two nonzero terms in the Laurent expansion of

$$f(z) = \tan\, z \qquad \text{about } z = \pi/2$$

Solution

Taking $z = \pi/2 + u$, a convergent LS for $|u| < \pi$ is setup below

$$f(z) = \frac{\sin\,(\pi/2+u)}{\cos\,(\pi/2+u)} = -\frac{\cos\,u}{\sin\,u} = -\frac{1}{u}\left(\frac{1-u^2/2!+u^4/4!\ldots}{1-u^2/3!+u^4/5!\ldots}\right)$$

We approximate the fraction

$$\frac{1}{1-u^2/3!+u^4/5!\ldots} = \frac{1}{1-\underbrace{(u^2/3!-u^4/5!\ldots)}_{\text{common ratio}}} = 1+u^2/3!+\mathcal{O}(u^3)$$

The two non-zero terms in LS

$$f(z) = -\frac{1}{u}\left(1-\frac{u^2}{2!}\right)\left(1+\frac{u^2}{3!}\right) = -\frac{1}{u}+\frac{u}{3} = -\frac{1}{z-\pi/2}+\frac{z-\pi/2}{3}$$