Lecture 19: Function Singularities

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Recap

We learned to write a Laurent series for any $f(z)$ about z_0,

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z - z_0)^n, \qquad C_n = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, \mathrm{d}\xi$$

The coefficient C_{-1} is called the residue as only this term will contribute to a loop integral of $f(z)$ about z_0.

For f analytic at z_0, $C_n = 0 \; \forall \; n < 0$ —negative powers don't exist.

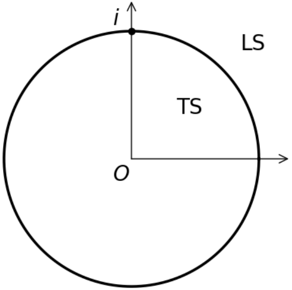
Lets look at some more worked examples.

Problem 1

Expand $f(z) = \dfrac{1}{1+z^2}$ in power series about $\underbrace{z=0}_{\text{regular}}$ and $\underbrace{z=i}_{\text{singular}}$

about $z = 0$

construct a circle,
$|z| = 1$



For $|z| < 1$, we derive a Taylor series

$$f(z) = \frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}$$

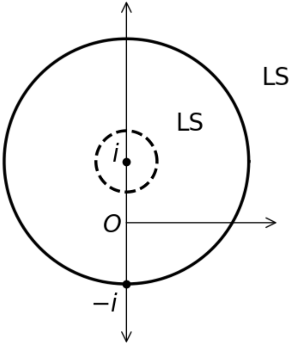
For $|z| > 1$, we derive a Laurent series

$$\begin{aligned} f(z) &= \frac{1}{z^2}\left[\frac{1}{1+(1/z)^2}\right] = \sum_{n=0}^{\infty}(-1)^n \frac{1}{z^{2n+2}} \\ C_{-1} &= 0 \quad \ldots \text{ Residue} \end{aligned}$$

Problem 1

— about $z = i$ —

construct two circles,
$|z-i| = a$ (small)
$|z-i| = 2$ (large)



$z = -i$ is a singularity

$f(z) = \dfrac{1}{(z+i)(z-i)}$ can be written

(i) in the annular region, $a < |z-i| < 2$ as

$$\begin{aligned} f(z) &= \frac{1}{(z-i)} \frac{1}{2i(1+\frac{z-i}{2i})} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^{n-1}}{(2i)^{n+1}} \quad \ldots C_{-1} = \frac{1}{2i} \end{aligned}$$

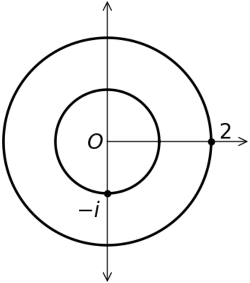
(ii) in the region $|z-i| > 2$ as

$$\begin{aligned} f(z) &= \frac{1}{(z-i)^2} \frac{1}{(1+\frac{2i}{z-i})} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(2i)^n}{(z-i)^{n+2}} \quad \ldots C_{-1} = 0 \end{aligned}$$

Problem 2

Expand the following function

$$f(z) = \frac{z}{(z-2)(z+i)} \qquad \text{about } z = 0$$



For $|z| < 1$, we set up a Taylor expansion

$$\begin{aligned} f(z) = \frac{1}{(2+i)}\left[\frac{2}{z-2} + \frac{i}{z+i}\right] &= \frac{1}{(2+i)}\left[\frac{-1}{1-z/2} + \frac{1}{1+z/i}\right] \\ &= \frac{1}{(2+i)}\sum_{n=0}^{\infty}\left[-\left(\frac{z}{2}\right)^n + \frac{(-1)^n z^n}{i^n}\right] \\ &= \frac{1}{(2+i)}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\left[(2i)^n - 1\right] \end{aligned}$$

Problem 2

In the annular region, $1 < |z| < 2$, we set up a Laurent expansion

$$f(z) = \frac{1}{(2+i)}\left[\frac{2}{z-2} + \frac{i}{z+i}\right] = \frac{1}{(2+i)}\left[\frac{-1}{1-z/2} + \frac{i}{z(1+i/z)}\right]$$

$$= \frac{1}{(2+i)}\sum_{n=0}^{\infty}\left[-\left(\frac{z}{2}\right)^{n} + (-1)^{n}\left(\frac{i}{z}\right)^{n+1}\right] \quad \ldots C_{-1} = \frac{i}{2+i}$$

In the region $|z| > 2$, we set up a Laurent expansion

$$f(z) = \frac{1}{(2+i)}\left[\frac{2}{z-2} + \frac{i}{z+i}\right] = \frac{1}{(2+i)}\left[\frac{2/z}{(1-2/z)} + \frac{i/z}{(1+i/z)}\right]$$

$$= \frac{1}{(2+i)}\sum_{n=0}^{\infty}\left[\left(\frac{2}{z}\right)^{n+1} + (-1)^{n}\left(\frac{i}{z}\right)^{n+1}\right]$$

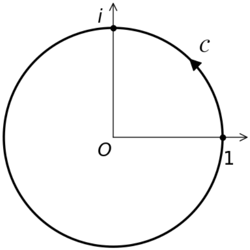
$$= \frac{1}{(2+i)}\sum_{n=0}^{\infty}\frac{2^{n+1} + (-1)^{n} i^{n+1}}{z^{n+1}} \quad \ldots C_{-1} = 1$$

Laurent Series to Solve Loop Integrals

- Laurent series about a singular point reveals the residue there
- This can be exploited to compute contour integrals directly
- We substantiate this argument with some examples next

Problem 3

Evaluate $\mathcal{I} = \oint_{\mathcal{C}} \frac{e^z}{z^3}\, \mathrm{d}z$



e^z is an entire, Taylor expanding it about $z = 0$

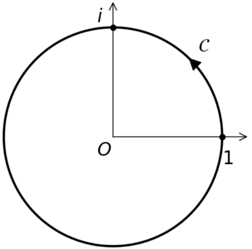
$$\mathcal{I} = \oint_{\mathcal{C}} \left[\frac{1}{z^3} + \frac{1}{z^2} + \underbrace{\frac{1}{2!z}}_{\text{survives}} + \frac{1}{3!} + \dots \right] \mathrm{d}z = \pi i$$

alternatively, from Cauchy's differential formula

$$f^n(z_0) = \frac{n!}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)}{(z - z_0)^{n+1}}\, \mathrm{d}z \implies \mathcal{I} = \frac{2\pi i}{2!} \frac{\mathrm{d}^2}{\mathrm{d}z^2} e^z \bigg|_{z=0} = \pi i$$

Problem 4

Evaluate $\mathcal{I} = \oint_{\mathcal{C}} e^{1/z} \, \mathrm{d}z$



$e^{1/z}$ has a singularity at $z = 0$

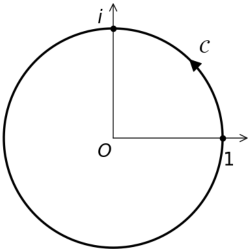
By mapping $z = 1/u$ we can Taylor expand at $z = \infty$ ($u = 0$)

$$e^u = 1 + u + \frac{u^2}{2!} + \mathcal{O}(u^3) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \mathcal{O}(1/z^3)$$

This is a Laurent series with residue $C_{-1} = 1$ yielding

$$\mathcal{I} = 2\pi i$$

Problem 5



Evaluate $\mathcal{I} = \oint_{\mathcal{C}} \frac{1}{z^2 \sin z} \, dz$

Note that $\frac{1}{z^2 \sin z}$ has singularities at $z = n\pi$ with $n \in \mathbb{Z}$

To compute $\mathcal{I}$, we set up a Laurent series of

$$\frac{1}{z^3} \frac{z}{\sin z} = \frac{1}{z^3} \frac{1}{1 - \underbrace{(z^2/3! + \mathcal{O}(z^4))}_{\text{common ratio}}} = \frac{1}{z^3} \left(1 + z^2/3! + \mathcal{O}(z^4)\right)$$

which always <u>converges</u> inside $\mathcal{C}$. The residue $C_{-1} = 1/3!$ gives

$$\mathcal{I} = \pi i/3$$

Singularities of Complex Functions

If a complex function can be written as

$$f(z) = \frac{\phi(z)}{(z - z_0)^m}$$

where $m \in \mathbb{N}^+$ and ϕ is analytic with $\phi(z_0) \neq 0$. Then f is said to have an isolated singularity at z_0 —characterized as a pole of order m. For $m = 1$, the singularity is just referred to as a simple pole.

Strength of Singularities

Since $\phi(z)$ is Taylor expandable at z_0, we rewrite the Laurent series

$$f(z) = \frac{1}{(z-z_0)^m} \sum_{n=0}^{\infty} \frac{1}{n!} \phi^n(z_0)(z-z_0)^n = \sum_{n=-m}^{\infty} C_n (z-z_0)^n$$

The coefficient of the greatest negative power $C_{-m} = \phi(z_0)$ is defined as the strength of the m^{th} order pole at z_0.

The residue in such cases is $C_{-1} = \dfrac{1}{(m-1)!} \phi^{m-1}(z_0)$.

I will illustrate these concepts with a worked example next.

Example

Describe the singularities of the function

$$f(z) = \frac{z^2 - 2z + 1}{z(z+1)^3}$$

Solution

$f(z)$ has a simple pole at $z = 0$ and a triple pole at $z = -1$

For the strength of simple pole at $z = 0$, we rewrite

$$f(z) = \frac{\phi(z)}{z} \quad \text{to yield } C_{-1} = \phi(0) = 1$$

For the strength of triple pole at $z = -1$, we rewrite

$$f(z) = \frac{\phi(z)}{(z+1)^3} \quad \text{to yield } C_{-3} = \phi(-1) = -4$$