Lecture 20: Residue Theorem

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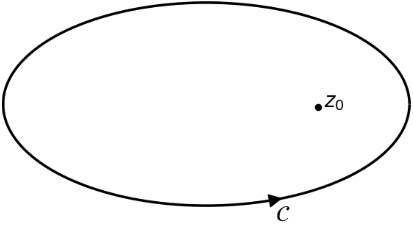
Cauchy's Residue Theorem

Let $f(z)$ be a function with an isolated singularity z_0 inside some $\mathcal{C}$



On the contour $\mathcal{C}$, we can write

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z - z_0)^n$$

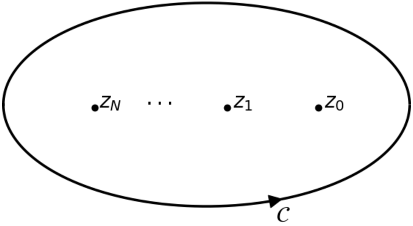
From which the integral

$$\boxed{\oint_{\mathcal{C}} f(z)\,\mathrm{d}z = 2\pi i \; C_{-1}}$$

Thus, loop integrals become very easy if we have the Laurent series

Generalization

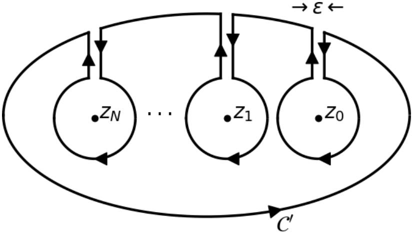
Including a finite number of isolated singularities inside $\mathcal{C}$



$$\oint_{\mathcal{C}} f(z)\, \mathrm{d}z = ?$$

Contour Deformation

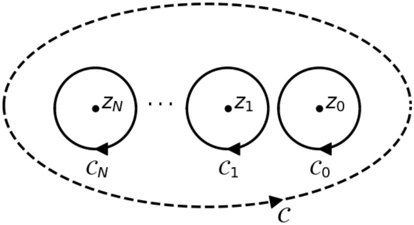
By making cuts of width ϵ, we form a new closed contour $\mathcal{C}'$



$$\oint_{\mathcal{C}'} f(z)\, \mathrm{d}z = 0$$

As all singularities are outside $\mathcal{C}'$

Stitch & Close



Taking $\epsilon \to 0$

$$\oint_{\mathcal{C}} f(z)\,\mathrm{d}z + \underbrace{\oint_{-\mathcal{C}_0} f(z)\,\mathrm{d}z + \oint_{-\mathcal{C}_1} f(z)\,\mathrm{d}z + \ldots \oint_{-\mathcal{C}_N} f(z)\,\mathrm{d}z}_{\text{negatively oriented}} = 0$$

$$\boxed{\oint_{\mathcal{C}} f(z)\,\mathrm{d}z = \oint_{\mathcal{C}_0} f(z)\,\mathrm{d}z + \oint_{\mathcal{C}_1} f(z)\,\mathrm{d}z + \ldots \oint_{\mathcal{C}_N} f(z)\,\mathrm{d}z = 2\pi i \sum_{j=0}^{N} \mathrm{res}_j}$$

Worked Examples

Evaluate $\mathcal{I} = \frac{1}{2\pi i} \oint_{\mathcal{C}} z\, e^{1/z}\, \mathrm{dz}$, where $\mathcal{C}$ is the circle $|z| = 1$

Solution

$z\, e^{1/z}$ is singular at $z = 0$ with a Laurent expansion nearby

$$z\, e^{1/z} = z + 1 + \frac{1}{2!z} + \frac{1}{3!z^2} + \ldots$$

with all negative powers of z. Thus $z = 0$ is an <u>essential singularity</u>
Reading the residue as $C_{-1} = 1/2$, the integral becomes

$$\mathcal{I} = C_{-1} = 1/2$$

Worked Examples

Evaluate $\mathcal{I} = \oint_{\mathcal{C}} \frac{z+2}{z(z+1)}\, \mathrm{d}z$, where $\mathcal{C}$ is the circle $|z| = 2$

Solution

Noting the two simple poles, namely at $z = (0, -1)$ we rewrite

$$\frac{z+2}{z(z+1)} = \frac{2}{z} - \frac{1}{z+1}$$

At $z = 0$, only $\frac{2}{z}$ will leave a residue $= 2$

At $z = -1$, only $\frac{-1}{z+1}$ will leave a residue $= -1$

Thus the integral is

$$\mathcal{I} = 2\pi i\,(2-1) = 2\pi i$$

Worked Example

Evaluate $\mathcal{I} = \dfrac{1}{2\pi i} \oint_{\mathcal{C}} \dfrac{3z+1}{z(z-1)^3}\, \mathrm{d}z$, where $\mathcal{C}$ is the circle $|z| = 2$

Solution

Two singularities, a simple pole at $z = 0$ and a triple pole at $z = 1$

Recalling that the residue due to the m^{th} order pole at z_0 is

$$C_{-1} = \frac{1}{(m-1)!} \phi^{(m-1)}(z_0)$$

Residue at $z = 0$, is $\left[\dfrac{3z+1}{(z-1)^3} \right]_{z=0} = -1$

Residue at $z = 1$, is $\dfrac{1}{2!} \dfrac{\mathrm{d}^2}{\mathrm{d}z^2} \left[\dfrac{3z+1}{z} \right]_{z=1} = 1$

The integral

$$\mathcal{I} = -1 + 1 = 0$$

Worked Example

Evaluate $\mathcal{I} = \dfrac{1}{2\pi i} \oint_{\mathcal{C}} \cot z \, \mathrm{d}z$, where $\mathcal{C}$ is the circle $|z| = 1$

Solution

The function $\cot z$ has singularities at $z = n\pi$ with $n \in \mathbb{Z}$

Inside the circle $\mathcal{C}$, we only have simple pole $z = 0$ (below)

$$\cot z = \frac{\cos z}{\sin z} = \frac{1}{z} \, \frac{\cos z}{\sin z/z} = \frac{\phi(z)}{z}$$

with $\phi(z)$ analytic inside $\mathcal{C}$ and $\phi(0) = 1 \neq 0$

The residue at the simple pole $z = 0$ is $\phi(0) = 1$ and the integral

$$\mathcal{I} = 1$$

Worked Example

Evaluate

$$\mathcal{I} = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{\phi(z)}{(az - z_0)} \, \mathrm{d}z \quad a \in \mathbb{R}$$

$\phi(z)$ is analytic inside $\mathcal{C}$ that encloses z_0/a, with $\phi(z_0/a) \neq 0$

Solution

For the simple pole at $z = z_0/a$, we rewrite

$$\frac{\phi(z)}{(az - z_0)} = \frac{\phi(z)}{a(z - z_0/a)}$$

that leaves the residue,

$$C_{-1} = \frac{\phi(z_0/a)}{a} = \mathcal{I}$$

Worked Example

Obtain $\mathcal{I} = \dfrac{1}{2\pi i} \oint_{\mathcal{C}} \dfrac{z+1}{2z^3 - 3z^2 - 2z}\, \mathrm{d}z$, with $\mathcal{C}$ as the circle $|z| = 1$

Solution

We rewrite

$$\frac{z+1}{2z^3 - 3z^2 - 2z} = \frac{z+1}{2z(z-2)(z+1/2)}$$

and read $z = 0, 2, -1/2$ as the three simple poles of this function

We will discount the pole at 2 as it is outside $\mathcal{C}$

Residue at $z = 0$ is $\left[\dfrac{z+1}{2(z-2)(z+1/2)}\right]_{z=0} = -\dfrac{1}{2}$

Residue at $z = -\dfrac{1}{2}$ is $\left[\dfrac{z+1}{2z(z-2)}\right]_{z=-1/2} = \dfrac{1}{5}$

$$\mathcal{I} = \frac{-1}{2} + \frac{1}{5} = \frac{-3}{10}$$