Lecture 21: Solving Definite Integrals

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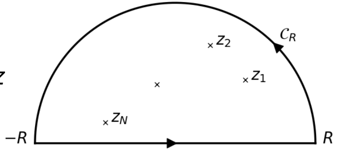
Definite Integrals

- To solve convergent real integrals of the form

$$\mathcal{I} = \int_{-\infty}^{\infty} f(x)\, \mathrm{d}x$$

- We consider instead the integral

$$\oint_{\mathcal{C}} f(z)\, \mathrm{d}z = \int_{-R}^{R} f(x)\, \mathrm{d}x + \int_{\mathcal{C}_R} f(z)\, \mathrm{d}z$$



Singularities of $f(z)$ inside $\mathcal{C}$

- LHS is computed from the residue theorem
- As $R \to \infty$, LHS does not change but the

  integral on the real axis becomes $\mathcal{I}$, our target!

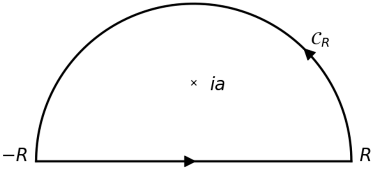
  integral on the arc vanishes asymptotically (Jordan's lemma)

Evaluate the integral $\mathcal{I} = \int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2} \quad a^2 > 0$

Solution

Define a function $f(z)$

$$\frac{1}{(z^2+a^2)^2} = \frac{1}{(z+ia)^2(z-ia)^2}$$



Consider the closed loop $\mathcal{C}$

Double pole at $z = ia$ has a residue $\frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{(z+ia)^2}\Big|_{z=ia} = \frac{1}{4ia^3}$

$$\frac{\pi}{2a^3} = \oint_{\mathcal{C}} f(z)\mathrm{d}z = \int_{-R}^{R} \frac{\mathrm{d}x}{(x^2+a^2)^2} + \int_0^\pi \underbrace{\frac{iRe^{i\theta}}{(R^2e^{2i\theta}+a^2)^2}}_{\text{scales as } 1/R^3} \mathrm{d}\theta$$

As $R \to \infty$ $\quad \frac{\pi}{2a^3} = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2+a^2)^2} = 2\mathcal{I}$

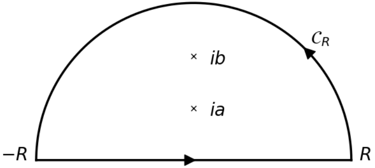
Worked Examples

Evaluate the integral $\mathcal{I} = \int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)(x^2+b^2)} \quad a^2, b^2 > 0$

Solution

Defining the function

$$f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)} = \frac{1}{(z+ia)(z-ia)(z+ib)(z-ib)}$$



Consider the closed loop $\mathcal{C}$

Residues at the simple poles at ia and ib yield

$$\oint_{\mathcal{C}} f(z)\,\mathrm{d}z = \frac{\pi}{ab(a+b)} = \int_{-R}^{R} \frac{\mathrm{d}x}{(x^2+a^2)(x^2+b^2)} + \int_{\mathcal{C}_R} \underbrace{f(z)\,\mathrm{d}z}_{\sim R^{-3}}$$

As $R \to \infty$ $\quad \frac{\pi}{ab(a+b)} = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2+a^2)(x^2+b^2)} = 2\mathcal{I}$

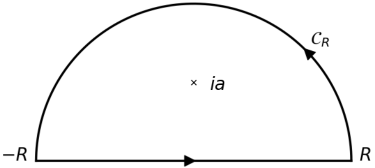
Problem

Evaluate the integral $\mathcal{I} = \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2}\, \mathrm{d}x \quad a^2 > 0$

Solution

Take the function

$$f(z) = \frac{z\ e^{iz}}{z^2 + a^2} = \frac{z\ e^{iz}}{(z + ia)(z - ia)}$$



Consider the closed loop $\mathcal{C}$

The residue at simple pole $z = ia$ yields

$$\oint_{\mathcal{C}} \frac{z\ e^{iz}}{z^2 + a^2}\, \mathrm{d}z = \frac{i\pi}{e^a} = \int_{-R}^{R} \frac{x\ e^{ix}}{x^2 + a^2}\, \mathrm{d}x + \int_0^{\pi} \underbrace{\left[\frac{iR^2 e^{2i\theta} e^{iR\cos\theta - R\sin\theta}}{R^2 e^{2i\theta} + a^2}\right]}_{\sim e^{-R}} \mathrm{d}\theta$$

As $R \to \infty$, the arc integral vanishes

$$\frac{i\pi}{e^a} = \int_{-\infty}^{\infty} \frac{x\ e^{ix}}{x^2 + a^2}\, \mathrm{d}x \implies \boxed{\frac{\pi}{e^a} = \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2}\, \mathrm{d}x = \mathcal{I}}$$

Food for Thought

- On comparing the real parts on both sides

$$0 = \int_{-\infty}^{\infty} \frac{x \cos x}{x^2 + a^2} \, \mathrm{d}x \quad \ldots \text{trivial since integrand is odd}$$

- The function

$$f(z) = \frac{z \, e^{-iz}}{z^2 + a^2}$$

with $\mathcal{C}$ in the $y < 0$ plane enclosing $z = -ia$ also works!

- Note that

$$f(z) = \frac{z \sin z}{z^2 + a^2}$$

is a bad choice as $|\sin z|$ diverges in the limit $z \to \infty$

Application in Probability Theory

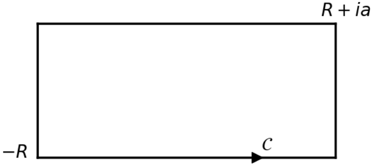
Integrate the complex Gaussian distribution on the real axis

$$\mathcal{I} = \int_{-\infty}^{\infty} e^{-(x+ia)^2}\, \mathrm{d}x \qquad a \in \mathbb{R}$$

Solution

Take the entire

$$f(z) = e^{-z^2}$$



and the loop $\mathcal{C}$ such that

$$0 = \oint_{\mathcal{C}} e^{-z^2}\, \mathrm{d}z = \int_{-R}^{R} e^{-x^2}\, \mathrm{d}x + \int_{R}^{-R} e^{-(x+ia)^2}\, \mathrm{d}x + \mathcal{I}_R$$

$$\text{with } \mathcal{I}_R = \int_0^a [e^{-(R+iy)^2} - e^{-(-R+iy)^2}] i \mathrm{d}y$$

$$= i\, e^{-R^2} \int_0^a [e^{-(1+iy/R)^2} - e^{-(-1+iy/R)^2}] \mathrm{d}y \longrightarrow 0 \ (R \to \infty)$$

continuing . . .

Thus we arrive at,

$$0 = \underbrace{\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x}_{\sqrt{\pi}} + \underbrace{\int_{\infty}^{-\infty} e^{-(x+ia)^2}\,\mathrm{d}x}_{-\mathcal{I}}$$

Yielding a result that is independent of a

$$\boxed{\mathcal{I} = \int_{-\infty}^{\infty} e^{-(x+ia)^2}\,\mathrm{d}x = \int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}}$$

Useful byproducts emerge on comparing both sides of above result!

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-x^2}\cos 2ax\,\mathrm{d}x &= \sqrt{\pi}\,e^{-a^2} \\ \int_{-\infty}^{\infty} e^{-x^2}\sin 2ax\,\mathrm{d}x &= 0 \end{aligned}$$

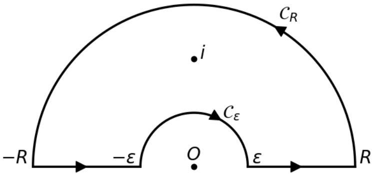
Problem

Evaluate $\mathcal{I} = \int_0^\infty \frac{\sin\, x}{x(x^2+1)}\,\mathrm{d}x$

Solution

Take the function

$$f(z) = \frac{e^{iz}}{z(z^2+1)} = \frac{e^{iz}}{z(z+i)(z-i)}$$



and the loop $\mathcal{C}$ such that

only the simple pole $z = i$ contributes to the loop integral

$$\oint_{\mathcal{C}} \frac{e^{iz}}{z(z^2+1)}\,\mathrm{d}z = -\frac{i\pi}{e} = \int_{x=-R}^{-\epsilon} + \int_{\mathcal{C}_\epsilon} + \int_{x=\epsilon}^{R} + \int_{\mathcal{C}_R} f(z)\,\mathrm{d}z$$

continuing...

On $\mathcal{C}_R$ choosing $z = Re^{i\theta}$

$$f(z)\,\mathrm{d}z \to \frac{e^{iR\cos\,\theta}\;e^{-R\sin\,\theta}}{R^2} \to 0 \text{ as } R \to \infty.$$

On $\mathcal{C}_\epsilon$ choosing $z = \epsilon e^{i\theta}$

$$\int_{\mathcal{C}_\epsilon} f(z)\,\mathrm{d}z = \int_\pi^0 \frac{e^{i\epsilon e^{i\theta}}\;i\;\epsilon\;e^{i\theta}}{\epsilon\;e^{i\theta}\;(\epsilon^2\;e^{i2\theta}+1)}\mathrm{d}\theta \to -i\pi \text{ as } \epsilon \to 0.$$

Applying these limits leads to,

$$-\frac{i\,\pi}{e} = \int_{-\infty}^{\infty} \frac{e^{ix}}{x\,(x^2+1)}\mathrm{d}x - i\,\pi \Rightarrow \boxed{\int_{-\infty}^{\infty} \frac{\sin\,x}{x\,(x^2+1)}\mathrm{d}x} = \pi\left(1-\frac{1}{e}\right).$$