Lecture 22: Fourier Series

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Fourier Series

Applications in

- Electrical engineering
- Vibration analysis
- Acoustics, Optics & Signal Processing
- Quantum Mechanics (particle in a box)

Formalism

Consider a periodic function $f(\theta)$ with $-\pi \leq \theta < \pi$

One can write an expansion for

$$f(\theta) = \underbrace{\frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n \cos n\theta + B_n \sin n\theta)}_{\text{Fourier Series}}$$

The coefficients can be recovered as

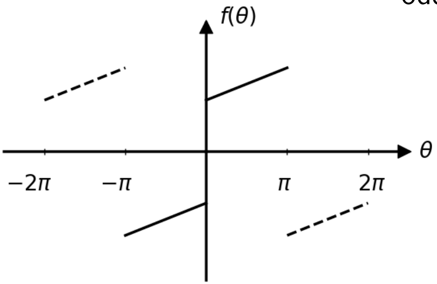
$$\begin{aligned} A_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos m\theta \, d\theta \\ B_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin m\theta \, d\theta \end{aligned}$$

from the orthogonality of the basis functions $\cos n\theta$ and $\sin n\theta$

Symmetry Arguments

Symmetry of $f(\theta)$ will dictate the form of the Fourier expansion

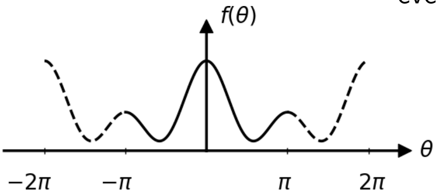
odd f



cosines disappear, $A_n = 0$

$$f(\theta) = \sum_{n=1}^{\infty} B_n \sin n\theta$$

even f

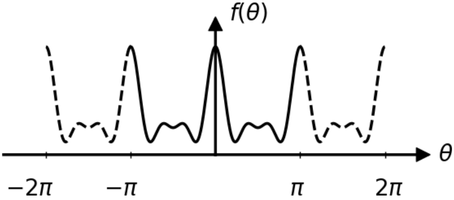


sines disappear, $B_n = 0$

$$f(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos n\theta$$

Symmetry Arguments

even f symmetric about $\pi/2$



sines disappear, $B_n = 0$
odd cosines disappear, $A_{2n+1} = 0$

$$f(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_{2n} \cos 2n\theta$$

More interesting cases can emerge! Let your imaginations roll

Worked Example

Derive the Fourier series for the function

$$f(\theta) = \begin{cases} +1 & 0 < \theta < \pi \\ -1 & \pi < \theta < 2\pi \end{cases}$$

Solution

S1: f is an odd function, so all cosines disappear ($A_n = 0$)

S2: f is symmetric about $\pi/2$, so even sines disappear ($B_{2n} = 0$)

Evaluating B_n for odd n,

$$B_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta) \sin\, n\theta \,\mathrm{d}\theta = \frac{1}{\pi}\left[\int_0^{\pi} \sin\, n\theta \,\mathrm{d}\theta - \int_{\pi}^{2\pi} \sin\, n\theta \,\mathrm{d}\theta\right] = \frac{4}{n\pi}$$

Thus yielding the Fourier series

$$\boxed{f(\theta) = \frac{4}{\pi}\left(\sin\,\theta + \frac{\sin\,3\theta}{3} + \frac{\sin\,5\theta}{5} + \dots\right)}$$

Possible Application

One can derive convergent sum for an infinite series. For eg. in

$$f(\theta) = \frac{4}{\pi}\left(\sin\,\theta + \frac{\sin\,3\theta}{3} + \frac{\sin\,5\theta}{5} + \dots\right)$$

putting $\theta = \pi/2$ gives

$$\frac{\pi}{4} = \underbrace{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots}_{\text{infinite series}}$$

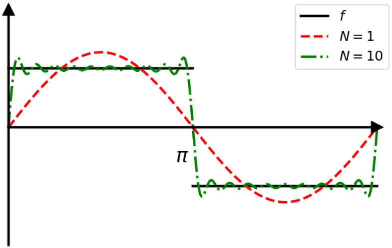
Discontinuities in $f(\theta)$

Fourier series does'nt converge at the discontinuities of $f(\theta)$

-Gibbs Phenomenon

Look at the N- partial sums

$$f(\theta) = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{4}{\pi} \frac{\sin\,(2n-1)\theta}{2n-1}$$



At $\theta = n\pi$, FS does'nt converge to f even as $N \to \infty$!

Worked Example

Work out the Fourier series for the function

$$f(\theta) = \cos\, k\theta \quad (-\pi < \theta < \pi) \qquad k \in \mathbb{R}$$

Solution

S1: Clearly $f(\theta)$ is even, so only cosine terms present i.e $B_n = 0$

$$A_n = \frac{2}{\pi}\int_0^\pi \cos\, k\theta \cos\, n\theta \, \mathrm{d}\theta = \frac{1}{\pi}\int_0^\pi [\cos\,(k+n)\theta \; + \cos\,(k-n)\theta]\, \mathrm{d}\theta$$

$$= \frac{1}{\pi}\left(\frac{\sin\,(k+n)\theta}{k+n}\Bigg|_0^\pi + \frac{\sin\,(k-n)\theta}{k-n}\Bigg|_0^\pi \right) = \frac{(-1)^n\, 2k \sin\, k\pi}{\pi(k^2-n^2)}$$

yielding the Fourier series

$$\boxed{\cos\, k\theta = \frac{2k \sin\, k\pi}{\pi}\left(\frac{1}{2k^2} - \frac{\cos\,\theta}{k^2-1} + \frac{\cos\, 2\theta}{k^2-4} - \dots\right)}$$

Problem

Derive and sketch the function $f(\theta)$ whose Fourier series is given as

$$\cos\theta + \frac{\cos 3\theta}{9} + \frac{\cos 5\theta}{25} + \dots$$

Solution

S1: f must be an even function

S2: f must be anti-symmetric at $\pi/2$: $f(\pi/2+\theta) = -f(\pi/2-\theta)$

From **S1**, we work out for odd n

$$A_n = \frac{2}{\pi}\int_0^\pi f(\theta)\cos n\theta \, d\theta = -\frac{2}{\pi n}\int_0^\pi f'(\theta)\sin n\theta \, d\theta = \frac{1}{n^2}$$

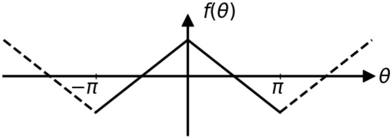
Possible with $f'(\theta) = -\frac{\pi}{4}$, giving $f(\theta) = -\frac{\pi}{4}\theta + B \quad (0 \le \theta < \pi)$

The constant B can be fixed from **S2**,

$$-\frac{\pi}{4}\left(\frac{\pi}{2}+\theta\right) + B = \frac{\pi}{4}\left(\frac{\pi}{2}-\theta\right) - B \implies \boxed{B = \frac{\pi^2}{8}}$$

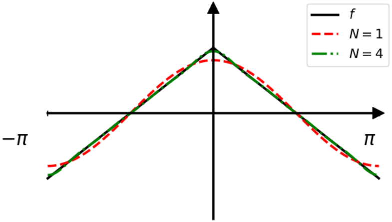
Sketching $f(\theta)$

$$f(\theta) = \begin{cases} -\pi\theta/4 + \pi^2/8, & 0 \le \theta < \pi \\ \pi\theta/4 + \pi^2/8, & -\pi \le \theta < 0 \end{cases}$$



Look at the N- partial sums

$$f(\theta) = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{\cos(2n-1)\theta}{(2n-1)^2}$$



FS converges everywhere quickly to f. “No Gibbs phenomenon”

Application in Convergent Sums

For any $\theta \in [-\pi, \pi]$, we can compute a convergent sum

For eg., plugging $\theta = 0$, we get

$$\frac{\pi^2}{8} = 1 + \frac{1}{9} + \frac{1}{25} + \dots$$