Lecture 27: Worked Examples

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Table of Laplace Transforms

$f(x)$	$\mathcal{L}[f(x)] = F(s)$	Convergence Condition
1	$\dfrac{1}{s}$	Re $s > 0$
$\delta(x - x_0) \quad (x > x_0)$	e^{sx_0}	
$\sin \lambda x$	$\dfrac{\lambda}{s^2 + \lambda^2}$	Re $s > 0$
$\cos \lambda x$	$\dfrac{s}{s^2 + \lambda^2}$	Re $s > 0$
x^n	$\dfrac{n!}{s^{n+1}}$	Re $s > 0$
$e^{-\lambda x}$	$\dfrac{1}{s + \lambda}$	Re $(s + \lambda) > 0$

Worked Example

Derive the function $f(x)$ whose Laplace transform is given as

$$\mathcal{L}[f(x)] = F(s) = \frac{1}{(s^2+1)(s-1)}$$

Solution

Method 1: We will use the table of transforms

$$F(s) = \frac{A}{(s+i)} + \frac{B}{(s-i)} + \frac{C}{(s-1)}$$

$$\text{with } A = \frac{1}{2i(i+1)} \qquad B = \frac{1}{2i(i-1)} \qquad C = \frac{1}{2}$$

$$f(x) = \mathcal{L}^{-1}[F(s)] = A \underbrace{e^{-ix}}_{\text{Re }(s+i)>0} + B \underbrace{e^{ix}}_{\text{Re }(s-i)>0} + C \underbrace{e^{x}}_{\text{Re }(s-1)>0}$$

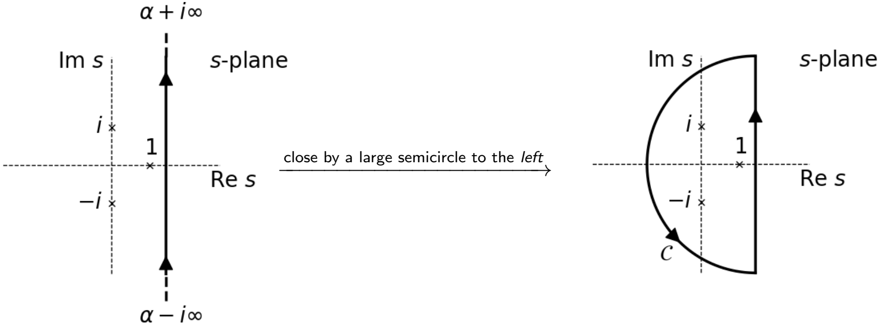
$$\boxed{f(x) = Ae^{-ix} + Be^{ix} + Ce^{x} \qquad (\text{Re } s > 1)}$$

Taking the bull by its horns...

Method 2: "Bromwich integral"

$$f(x) = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha+\infty} e^{sx}\ F(s)\ \mathrm{d}s = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha+\infty} \frac{e^{sx}}{(s+i)(s-i)(s-1)}\ \mathrm{d}s$$

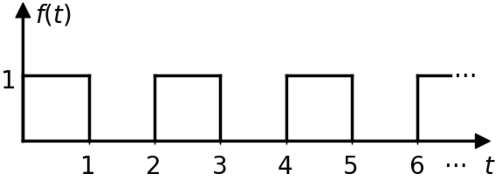
We must take $\alpha > 1$ as $F(s)$ is analytic only in the plane Re $(s) > 1$



$$\frac{1}{2\pi i}\oint_{\mathcal{C}} e^{sx}\ F(s)\ \mathrm{d}s = \sum_{z_0 = i, -i, 1} \text{Res}\ [z_0] = \frac{e^{-ix}}{2i(i+1)} + \frac{e^{ix}}{2i(i-1)} + \frac{e^{x}}{2} = f(x)$$

Worked Example

Find the Laplace transform of the square wave triggered at $t = 0$



Solution

$$
\begin{aligned}
F(s) = \int_0^\infty e^{-st}\, f(t)\, \mathrm{d}t &= \int_0^1 e^{-st}\, \mathrm{d}t + \int_2^3 e^{-st}\, \mathrm{d}t + \dots \\
&= \frac{1}{s}\left(1 - e^{-s} + e^{-2s} - e^{-3s} + \dots\right) \\
&= \frac{1}{s(1 + e^{-s})} \qquad \dots |e^{-s}| < 1 \equiv \mathrm{Re}\ s > 0
\end{aligned}
$$

Ordinary Differential Equations (ODE)

Laplace transforms can be used to solve ODEs. For eg.,

$$\begin{aligned}\mathcal{L}[f'(x)] = \int_0^\infty e^{-sx}\ f'(x)\,\mathrm{d}x &= e^{-sx}\ f(x)\Big|_0^\infty + s\int_0^\infty e^{-sx}\ f(x)\,\mathrm{d}x \\ &= s\ \mathcal{L}[f(x)] - f(0)\end{aligned}$$

Similarly,

$$\begin{aligned}\mathcal{L}[f''(x)] = s\ \mathcal{L}[f'(x)] - f'(0) &= s(s\ \mathcal{L}[f(x)] - f(0)) - f'(0) \\ &= s^2\ \mathcal{L}[f(x)] - s\ f(0) - f'(0)\end{aligned}$$

Thus to solve an n^{th} order ODE, we need $\underbrace{f(0), f^{(1)}(0)\ldots f^{(n-1)}(0)}_{\text{boundary (or initial) conditions}}$

Three radioactive nuclei decay successively in series

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots$$

$$\begin{aligned} \frac{dN_1}{dt} &= -\lambda_1 N_1 \\ \frac{dN_2}{dt} &= -\lambda_2 N_2 + \lambda_1 N_1 \\ \frac{dN_3}{dt} &= -\lambda_3 N_3 + \lambda_2 N_2 \end{aligned}$$

If $N_1(0) = N, N_2(0) = 0, N_3(0) = n$, find $N_3(t)$

Solution

Taking Laplace transforms of these ODEs,

$$\begin{aligned} s\,\mathcal{L}[N_1] - N_1(0) &= s\,\mathcal{L}[N_1] - N = -\lambda_1\,\mathcal{L}[N_1] \\ s\,\mathcal{L}[N_2] - N_2(0) &= s\,\mathcal{L}[N_2] = -\lambda_2\,\mathcal{L}[N_2] + \lambda_1\,\mathcal{L}[N_1] \\ s\,\mathcal{L}[N_3] - N_3(0) &= s\,\mathcal{L}[N_3] - n = -\lambda_3\,\mathcal{L}[N_3] + \lambda_2\,\mathcal{L}[N_2] \end{aligned}$$

continued ...

$$\begin{aligned}
\mathcal{L}[N_1] &= \frac{N}{\lambda_1 + s} \\
\mathcal{L}[N_2] &= \frac{\lambda_1 N}{(\lambda_1 + s)(\lambda_2 + s)} \\
\mathcal{L}[N_3] &= \frac{1}{\lambda_3 + s}\left[n + \frac{\lambda_1 \lambda_2 N}{(\lambda_1 + s)(\lambda_2 + s)}\right]
\end{aligned}$$

Using tables, we invert the last

$$\boxed{N_3(t) = ne^{-\lambda_3 t} + \lambda_1 \lambda_2 N(Ae^{-\lambda_1 t} + Be^{-\lambda_2 t} + Ce^{-\lambda_3 t})}$$

with

$$\frac{1}{(\lambda_1 + s)(\lambda_2 + s)(\lambda_3 + s)} = \frac{A}{\lambda_1 + s} + \frac{B}{\lambda_2 + s} + \frac{C}{\lambda_3 + s}$$

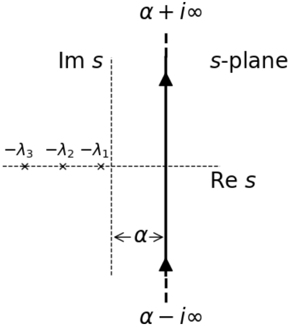
$$A = \frac{1}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)}, B = \frac{1}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)}, C = \frac{1}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)}$$

Using Bromwich integral

Recalling that

$$\mathcal{L}[N_3] = \frac{1}{\lambda_3 + s}\left[n + \frac{\lambda_1\lambda_2 N}{(\lambda_1 + s)(\lambda_2 + s)}\right]$$

$$N_3(t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{e^{st}}{\lambda_3 + s}\left[n + \frac{\lambda_1\lambda_2 N}{(\lambda_1 + s)(\lambda_2 + s)}\right] \mathrm{d}s$$



Closing the contour by a large semicircle on the *left*

$$N_3(t) = ne^{-\lambda_3 t} + \lambda_1\lambda_2 N(Ae^{-\lambda_1 t} + Be^{-\lambda_2 t} + Ce^{-\lambda_3 t})$$

with A, B, C defined as before