# Lecture 28: Convolution Theorems

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# Laplace Convolution

Let $f_1(x)$ and $f_2(x)$ be two arbitrary functions. Their convolution

$$(f_1 * f_2)(x) = \int_0^x f_1(x')\, f_2(x - x')\, \mathrm{d}x'$$

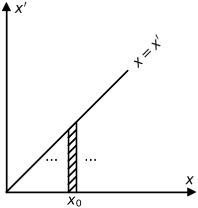
has the property

$$\underbrace{\mathcal{L}[(f_1 * f_2)(x)]}_{\text{Transform of the convolution}} = \underbrace{\mathcal{L}[f_1(x)]\, \mathcal{L}[f_2(x)]}_{\text{product of transforms}}$$

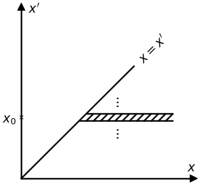
**Proof**

In what follows, we will give a proof by a geometrical insight!

# Area Swept




same area under $x = x'$




$$
\begin{aligned}
\mathcal{L}[(f_1 * f_2)(x)] &= \int_{x=0}^{\infty} e^{-sx} \int_{x'=0}^{x} f_1(x')\, f_2(x-x')\, \mathrm{d}x'\, \mathrm{d}x && \ldots(\text{left}) \\
&= \int_{x'=0}^{\infty} f_1(x') \underbrace{\int_{x=x'}^{\infty} e^{-sx}\, f_2(x-x')\, \mathrm{d}x}_{x-x'=u;\ \mathrm{d}x=\mathrm{d}u}\, \mathrm{d}x' && \ldots(\text{right}) \\
&= \int_{x'=0}^{\infty} f_1(x') \int_{u=0}^{\infty} e^{-s(x'+u)}\, f_2(u)\, \mathrm{d}u\, \mathrm{d}x' \\
&= \int_{x'=0}^{\infty} e^{-sx'}\, f_1(x')\, \mathrm{d}x' \int_{u=0}^{\infty} e^{-su}\, f_2(u)\, \mathrm{d}u \\
&= \mathcal{L}[f_1(x)]\, \mathcal{L}[f_2(x)]
\end{aligned}
$$

# Convolution is Commutative

For any two functions $f_1(x)$ and $f_2(x)$, prove that

$$(f_1 * f_2)(x) = (f_2 * f_1)(x)$$

**Proof**

We start from

$$\begin{aligned}
(f_1 * f_2)(x) &= \int_0^x f_1(x')\, f_2(x - x')\, \mathrm{d}x' \\
\ldots &\quad \text{set } x - x' = u;\ \mathrm{d}x' = -\mathrm{d}u \\
&= -\int_x^0 f_1(x-u)\, f_2(u)\, \mathrm{d}u \\
&= \int_0^x f_2(u)\, f_1(x-u)\, \mathrm{d}u \\
&= (f_2 * f_1)(x)
\end{aligned}$$

This result can be exploited in inversion problems

# Revisiting an Old Problem

Using the method of convolutions, invert the Laplace transform

$$\mathcal{L}[f(x)] = F(s) = \frac{1}{(s^2+1)(s-1)}$$

**Solution**

$$F(s) = \frac{1}{s^2+1}\,\frac{1}{s-1} \equiv F_1(s)\,F_2(s)$$

$$f_1(x) = \mathcal{L}^{-1}[F_1(s)] = \sin x \qquad f_2(x) = \mathcal{L}^{-1}[F_2(s)] = e^x$$

$$\begin{aligned} f(x) = \int_0^x f_1(x')\,f_2(x-x')\,\mathrm{d}x' &= \int_0^x \sin x'\, e^{x-x'}\,\mathrm{d}x' \\ &= \frac{e^x}{2i}\left[\frac{e^{(i-1)x'}}{i-1} + \frac{e^{-(i+1)x'}}{i+1}\right]_0^x \\ &= \frac{1}{2i}\left[\frac{e^{ix}}{i-1} + \frac{e^{-ix}}{i+1} + ie^x\right] \end{aligned}$$

# Fourier Convolution

Let $f_1(x)$ and $f_2(x)$ be two arbitrary functions. Their convolution

$$(f_1 * f_2)(x) = \int_{-\infty}^{\infty} f_1(x')\, f_2(x - x')\, \mathrm{d}x'$$

has the property

$$\underbrace{\mathcal{F}[(f_1 * f_2)(x)]}_{\text{Transform of the convolution}} \quad = \quad \underbrace{\tilde{f}_1(k)\, \tilde{f}_2(k)}_{\text{product of transforms}}$$

**Proof**

$$\begin{aligned}
\mathcal{F}[(f_1 * f_2)(x)] &= \int_{-\infty}^{\infty} e^{-ikx} \int_{-\infty}^{\infty} f_1(x')\, f_2(x - x')\, \mathrm{d}x'\, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} f_1(x') \underbrace{\left[\int_{-\infty}^{\infty} e^{-ikx}\, f_2(x - x')\, \mathrm{d}x\right]}_{x - x' = u;\ \mathrm{d}x = \mathrm{d}u} \mathrm{d}x' \\
&= \int_{-\infty}^{\infty} f_1(x')\, e^{-ikx'}\, \mathrm{d}x' \int_{-\infty}^{\infty} e^{-iku}\, f_2(u)\, \mathrm{d}u \\
&= \tilde{f}_1(k)\, \tilde{f}_2(k)
\end{aligned}$$

# Beautiful Corollary

$$\underbrace{(\tilde{f}_1 * \tilde{f}_2)(k)}_{\text{Convolution in } k\text{-space}} = \underbrace{\mathcal{F}[f_1(x)\ f_2(x)]}_{\text{Transform of product in } x\text{-space}} \quad 2\pi$$

**Proof**

$$\begin{aligned}
\text{RHS} &= 2\pi\ \mathcal{F}[f_1(x)\ f_2(x)] \\
&= 2\pi \int_{-\infty}^{\infty} e^{-ikx} f_1(x)\ f_2(x)\ \mathrm{d}x \\
&= \frac{2\pi}{(2\pi)^2} \int_{-\infty}^{\infty} e^{-ikx} \int_{-\infty}^{\infty} \tilde{f}_1(k')\ e^{ik'x}\ \mathrm{d}k' \int_{-\infty}^{\infty} \tilde{f}_2(k'')\ e^{ik''x}\ \mathrm{d}k''\ \mathrm{d}x \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}x\ \mathrm{d}k'\ \mathrm{d}k'' e^{-i(k-(k'+k''))x}\ \tilde{f}_1(k')\ \tilde{f}_2(k'') \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}k'\ \mathrm{d}k'' \delta(k-(k'+k''))\ \tilde{f}_1(k')\ \tilde{f}_2(k'') \\
&= \int_{-\infty}^{\infty} \mathrm{d}k'\ \tilde{f}_1(k')\ \tilde{f}_2(k-k') = (\tilde{f}_1 * \tilde{f}_2)(k) = \text{LHS}
\end{aligned}$$