Lecture 30: Physical Applications - II

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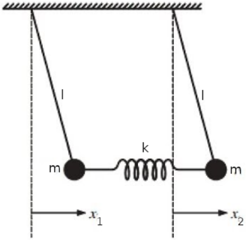
Coupled Pendulums

Consider the coupled pendulums shown below



If the initial conditions are

$$x_1(0) = 0, \quad x_2(0) = 0$$
$$\dot{x}_1(0) = v, \quad \dot{x}_2(0) = 0$$

Solve for $x_1(t), x_2(t)$

Solution

Governing (Newton's) equations are

$$m\ddot{x}_1 = -\frac{mg}{l}x_1 + k(x_2 - x_1) = -\frac{\partial U}{\partial x_1}$$
$$m\ddot{x}_2 = -\frac{mg}{l}x_2 + k(x_1 - x_2) = -\frac{\partial U}{\partial x_2}$$

Defining $\mathcal{L}[x_i(t)] = X_i(s)$, we take the Laplace transform of above

Laplace Transform

$$\begin{aligned} m(s^2X_1 - v) &= -\frac{mg}{l}\,X_1 + k(X_2 - X_1) \\ ms^2X_2 &= -\frac{mg}{l}\,X_2 + k(X_1 - X_2) \end{aligned}$$

Solving for X_1,

$$X_1(s) = \frac{v\left(s^2 + \frac{g}{l} + \frac{k}{m}\right)}{\left(s^2 + \frac{g}{l} + 2\frac{k}{m}\right)\left(s^2 + \frac{g}{l}\right)} = \frac{v}{2}\left(\frac{1}{s^2 + \frac{g}{l} + 2\frac{k}{m}} + \frac{1}{s^2 + \frac{g}{l}}\right)$$

Inverting,

$$\boxed{x_1(t) = \mathcal{L}^{-1}[X_1(s)] = \frac{v}{2}\left(\frac{\sin\,\omega_1 t}{\omega_1} + \frac{\sin\,\omega_2 t}{\omega_2}\right)}$$

with the angular frequencies $\omega_1 = \sqrt{\frac{g}{l} + 2\frac{k}{m}}$ and $\omega_2 = \sqrt{\frac{g}{l}}$

Limit of Weak Coupling

Notice that as

$$k \longrightarrow 0 \qquad x_1(t) \longrightarrow \frac{v}{\omega} \sin \omega t \qquad \omega = \sqrt{\frac{g}{l}}$$

we recover the case of a simple pendulum

Derivation of $x_2(t)$ is similar (left as exercise)

Physical Meaning of $\omega_{1,2}$

Close to mechanical equilibrium, the potential energy

$$U = U_0 + \frac{\partial U}{\partial x_1}\,\mathrm{d}x_1 + \frac{\partial U}{\partial x_2}\,\mathrm{d}x_2 + \sum_{ij=(1,2)} \frac{\partial^2 U}{\partial x_i \partial x_j}\,\mathrm{d}x_i\,\mathrm{d}x_j + \mathcal{O}(\mathrm{d}x^3)$$

$$U \approx \sum_{ij=(1,2)} \frac{\partial^2 U}{\partial x_i \partial x_j}\,\mathrm{d}x_i\,\mathrm{d}x_j \qquad \ldots \text{by setting } U_0 = 0$$

The matrix of second derivatives or the Hessian is

$$\mathcal{H} = \begin{bmatrix} \frac{\partial^2 U}{\partial x_1 \partial x_1} & \frac{\partial^2 U}{\partial x_1 \partial x_2} \\ \frac{\partial U}{\partial^2 x_2 \partial x_1} & \frac{\partial^2 U}{\partial x_2 \partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{mg}{l} + k & -k \\ -k & \frac{mg}{l} + k \end{bmatrix}$$

The eigenvalues of $\mathcal{H}$ are $\left(\frac{mg}{l} + 2k, \frac{mg}{l}\right) \equiv m\omega^2$

Yielding the **normal modes**, $\boxed{\omega_1 = \sqrt{\frac{g}{l} + 2\frac{k}{m}} \text{ and } \omega_2 = \sqrt{\frac{g}{l}}}$

Further Problems for this Chapter

Obtain the Fourier transform of the following functions

1. $e^{-|x|}$
2. $\dfrac{1}{x^2 + a^2}, \quad a^2 > 0$
3. $\dfrac{1}{(x^2 + a^2)^2}, \quad a^2 > 0$

Recover the functions with the following Laplace transforms

1. $\dfrac{1}{(s + \omega)^n}, \quad \omega > 0$
2. $\dfrac{s}{(s + \omega)^n}, \quad \omega > 0$
3. $\dfrac{1}{(s + \omega_1)(s + \omega_2)^2}, \quad \omega_1, \omega_2 > 0$
4. $\dfrac{1}{s^2(s^2 + \omega^2)}, \quad \omega > 0$