

Some important probability distributions.

(1) The normal (Gaussian) distribution:

$$p(x) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left(-\frac{(x-\lambda)^2}{2\sigma^2}\right)$$

The characteristic or the moment generating function is:

$$p(k) = \int_{-\infty}^{+\infty} dx e^{-ikx} p(x) = \int_{-\infty}^{+\infty} dx e^{-ikx} \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-\frac{(x-\lambda)^2}{2\sigma^2}}$$

$$= \frac{1}{(2\pi\sigma^2)^{1/2}} \int_{-\infty}^{+\infty} dx e^{-\frac{(x^2 + \lambda^2 - 2x\lambda + 2ikx\sigma^2)}{2\sigma^2}}$$

$$= \frac{1}{(2\pi\sigma^2)^{1/2}} \int_{-\infty}^{+\infty} dx e^{-\frac{(x-\lambda+ik\sigma^2)^2}{2\sigma^2}} \cdot e^{-\frac{\lambda^2}{2\sigma^2}} \cdot e^{\frac{(\lambda-ik\sigma^2)^2}{2\sigma^2}}$$

$$= \frac{1}{(2\pi\sigma^2)^{1/2}} \cdot e^{-ik\lambda - \frac{k^2\sigma^2}{2}} (2\pi\sigma^2)^{1/2}, \text{ using } \int_{-\infty}^{+\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

$$= e^{-ik\lambda - \frac{k^2\sigma^2}{2}}$$

∴ Cumulant generating function is

$$\ln p(k) = -ik\lambda - \frac{k^2\sigma^2}{2}$$

$$\therefore 1^{\text{st}} \text{ Cumulant: } \langle x \rangle_c = \left. \frac{\partial}{\partial(-ik)} \ln p(k) \right|_{k=0} = \lambda$$

$$\wedge 2^{\text{nd}} \text{ Cumulant: } \langle x^2 \rangle_c = \left. \frac{\partial^2}{\partial(-ik)^2} \ln p(k) \right|_{k=0} = \sigma^2$$

All higher cumulants are zero $\{ \langle x^n \rangle_c = 0, \text{ if } n > 2 \}$

"Thus the normal distribution is completely specified by its two first cumulants." In fact, it is the only distribution in nature to have only the first two cumulants.

Moments of the normal distribution are thus easily calculated:

$$\langle x \rangle = \lambda$$

$$\langle x^2 \rangle = \sigma^2 + \lambda^2$$

$$\langle x^3 \rangle = 3\sigma^2\lambda + \lambda^3$$

⋮

The normal distribution is one of the most important distributions in physics. The vanishing of all higher order cumulants beyond the order 3 implies that all graphical computations of moments require only the products of one point and two point propagators.

(2) Binomial distributions: Consider a random variable with two possible outcomes (say Head or Tail) A and B. The respective probabilities are p_A & p_B with the requirement: $p_A + p_B = 1$. Now, the probability that in N trials, the event A will occur exactly N_A times is

$$p_N(N_A) = \binom{N}{N_A} p_A^{N_A} p_B^{N-N_A}$$

No. of possible orderings of event A

Probability of exactly one such ordering.

Next, we look at the characteristic function of this distribution.

$$P_N(k) = \langle e^{-ikN_A} \rangle = \sum_{N_A=0}^N \frac{N!}{(N-N_A)! N_A!} p_A^{N_A} p_B^{(N-N_A)} e^{-ikN_A}$$

$$= (p_A e^{-ik} + p_B)^N \quad \left\{ \begin{array}{l} \text{Recalling the binomial expansion} \\ \text{of } (x+y)^N = \sum_{n=0}^N \binom{N}{n} x^n y^{N-n} \end{array} \right.$$

The cumulant generating function is thus given as,

$$\ln P_N(k) = N \ln(p_A e^{-ik} + p_B) = N \ln p_1(k)$$

$$\therefore p_1(k) = p_A e^{-ik} + p_B$$

Since $\ln p_1(k)$ is the cumulant generating function for a single trial, above relation tells us that the cumulants after N trials are simply N times the cumulants after the first trial.

$$\text{Also, in a single trial } \langle N_A^l \rangle = \left. \frac{\partial^l}{\partial (-ik)^l} p_1(k) \right|_{k=0} = p_A$$

The first two cumulants after N -trials are:

$$\langle N_A \rangle_c = \langle N_A \rangle = N p_A$$

$$\langle N_A^2 \rangle_c = \langle N_A^2 \rangle - \langle N_A \rangle^2 = N p_A p_B$$

proof: $\langle N_A^2 \rangle_c = N \frac{\partial^2}{\partial (-ik)^2} \ln p_i(k) \Big|_{k=0}$

$$= N \left[\frac{-1}{p_i(k)^2} \left(\frac{\partial p_i(k)}{\partial (-ik)} \right)^2 + \frac{1}{p_i(k)} \frac{\partial^2 p_i(k)}{\partial (-ik)^2} \right]_{k=0}$$

$$= N [-p_A^2 + p_A]$$

$$= N p_A p_B \quad \because p_A + p_B = 1.$$

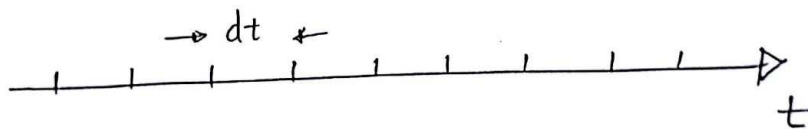
The binomial distribution is straightforwardly generalized to multinomial distribution, when the several outcomes $\{A, B, \dots, M\}$ occur with probabilities $\{p_A, p_B, \dots, p_M\}$. Thus probability of finding outcomes $\{N_A, N_B, \dots, N_M\}$ in a total of $N_A + N_B + \dots + N_M = N$ trials is

$$P_N(N_A, N_B, \dots, N_M) = \frac{N!}{N_A! N_B! N_C! \dots N_M!} p_A^{N_A} p_B^{N_B} \dots p_M^{N_M}$$

(3) Poisson distribution: A classic example of a Poisson process is radioactive β -decay. Observing the decay process of a radio-active material over a time interval T shows that,

(a) Probability of one and only one event (decay) in the time interval $[t, t+dt]$ is proportional to dt as $dt \rightarrow 0$

(b) Probabilities of events at different intervals are independent of each other.



The probability of having M events in the time interval T is given by a Poisson distribution.

- Divide entire interval T into N parts such that $dt = T/N$
 $N \gg 1$

- Probability of having an event in dt is $p = \alpha dt$

- Probability of having no event in dt is $q = 1 - p = 1 - \alpha dt$

As the probability to have more than one event in dt is too small to consider, the process is equivalent to a binomial one. Thus, the characteristic function is,

$$\begin{aligned} \phi(k) &= (pe^{-ik} + q)^n = \lim_{dt \rightarrow 0} \left(\alpha dt e^{-ik} + (1 - \alpha dt) \right)^{T/dt} \\ &= \left(1 + \alpha dt (e^{-ik} - 1) \right)^{T/dt} \quad \because p = \alpha dt \\ &\quad \quad \quad q = 1 - p = 1 - \alpha dt \end{aligned}$$

Using the result that for large n and small x $(1+x)^n \simeq e^{nx}$, we write

$$\phi(k) = e^{\alpha T (e^{-ik} - 1)}$$

To obtain Poisson PDF, we need to take the inverse transform of $\phi(k)$

$$\begin{aligned} \therefore f(x) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk e^{ikx} e^{\alpha T (e^{-ik} - 1)} \\ &= \frac{1}{2\pi} \cdot e^{-\alpha T} \cdot \int_{-\infty}^{+\infty} dk e^{ikx} e^{\alpha T e^{-ik}} \end{aligned}$$

$$p(x) = \frac{e^{-\alpha T}}{2\pi} \int_{-\infty}^{+\infty} dk e^{ikx} \sum_{M=0}^{\infty} \frac{(\alpha T e^{-ik})^M}{M!}$$

$$= \frac{e^{-\alpha T}}{2\pi} \sum_{M=0}^{\infty} \frac{(\alpha T)^M}{M!} \int_{-\infty}^{+\infty} dk e^{ik(x-M)}$$

Using the result $\frac{1}{2\pi} \int_{-\infty}^{+\infty} dk e^{ik(x-M)} = \delta(x-M)$
 Dirac Delta function

the above equation reduces to

$$p(M) = \frac{e^{-\alpha T} (\alpha T)^M}{M!}$$

"This is the probability of having M events in a time interval T during a Poisson process".

Clearly the cumulant generating function is

$$\ln p(k) = \alpha T (e^{-ik} - 1)$$

\therefore the m^{th} cumulant becomes,

$$\langle M^m \rangle_c = \frac{\partial^m}{\partial (-ik)^m} \ln p(k) \Big|_{k=0} = \alpha T$$

Thus all the cumulants have the same value. So,
the moments are obtained as

$$\langle M \rangle = \alpha T$$

$$\langle M^2 \rangle = \alpha T + (\alpha T)^2$$

$$\langle M^3 \rangle = \alpha T + 3(\alpha T)^2 + (\alpha T)^3$$