## Many random variables.

$S_{\vec{x}} = \{-\infty < x_1, x_2, \ldots x_N < +\infty\}$ is a set of outcomes in an N-dimensional space. For eg. describing the position and velocity of a gas particle requires six coordinates.

* **Joint PDF** $p(\vec{x})$: It is the probability density of an outcome in the volume element $d^N\vec{x} = \prod_{i=1}^{N} dx_i$
$= dx_1 dx_2 \ldots dx_N$

around the point $\vec{x} = \{x_1, x_2 \ldots x_N\}$

For eg. $p(\vec{x}) d^N\vec{x} = p(x, y, z, V_x, V_y, V_z)\, dx\, dy\, dz\, dV_x\, dV_y\, dV_z$

$$\equiv \text{prob}\big[E \in (x, x+dx) \;\&\; (y, y+dy) \;\&\; (z, z+dz) \;\& \\ (V_x, V_x+dV_x) \;\&\; (V_y, V_y+dV_y) \;\& \\ (V_z, V_z+dV_z)\big]$$

It is normalized : $\int p(\vec{x}) d^N\vec{x} = 1$

If N-random numbers are independent,

$$p(\vec{x}) = p(x_1) p(x_2) \ldots p(x_N) = \prod_{i=1}^{N} p(x_i)$$

**Unconditional PDF:** It describes the behavior of the subset of random variables, independent of the values of others. For eg. if only the velocity of a gas particle is required, then one integrates the joint PDF over all positions at a given value of velocity. Mathematically, this means:

$$p(\vec{v}) = \int d\vec{x}\, p(\vec{x}, \vec{v}) = \iiint dx\, dy\, dz\, p(x, y, z, v_x, v_y, v_z)$$

More generally:

$$p(x_1, x_2, \dots x_m) = \int \prod_{i=m+1}^{N} dx_i\, p(x_1, x_2, \dots x_N)$$

**Conditional PDF:** It describes the behavior of a subset of random variables, for a specified value of the others. For eg. the PDF of velocity of a particle at a particular location $\vec{x}$ is given as.

$$p(\vec{v}|\vec{x}) = \frac{p(\vec{x}, \vec{v})}{\mathcal{N}} \qquad \therefore p(\vec{v}|\vec{x}) \propto p(\vec{x}, \vec{v})$$

where $\mathcal{N} = \int d\vec{v}\, p(\vec{x}, \vec{v}) = p(\vec{x})$

$\mathcal{N}$ is the unconditional PDF for a particle at $\vec{x}$..

In general, the ~~un~~conditional PDF's are obtained from the Baye's theorem.

$$p(x_1, x_2, \ldots x_m / x_{m+1}, x_{m+2}, \ldots x_N) = \frac{p(x_1, x_2, \ldots x_N)}{p(x_{m+1}, x_{m+2}, \ldots, x_N)}$$

If the random numbers are independent, the unconditional PDF is equal to conditional PDF. We prove this for $N = 2$.

Conditional PDF : $p(x_1 | x_2) = \dfrac{p(x_1, x_2)}{p(x_2)}$

$\because$ $x_1$ and $x_2$ are independent,

$$p(x_1, x_2) = p(x_1)\, p(x_2)$$

$\therefore$ $p(x_1 | x_2) = p(x_1)$ which is the unconditional PDF.

*. The expectation value :

$$\langle F(\vec{x}) \rangle = \int d\vec{x}_N \, p(\vec{x}) \, F(\vec{x})$$

*. Joint characteristic function: It is the N-dimensional Fourier transform of the joint PDF.

$$\tilde{p}(\vec{k}) = \int \cdots \int dx_1 \, dx_2 \ldots dx_N \, e^{-ik_1x_1} \, e^{-ik_2x_2} \ldots e^{-ik_Nx_N} \, p(x_1, x_2, \ldots x_N)$$

$$= \int d^N\vec{x} \, \exp\left(-i\sum_{j=1}^{N} k_j x_j\right) p(\vec{x})$$

$$= \left\langle \exp\left(-i\sum_{j} k_j x_j\right) \right\rangle$$

∴ the joint moments and cumulants are given as -

$$\langle x_1^{n_1} x_2^{n_2} \ldots x_N^{n_N} \rangle = \left[\frac{\partial}{\partial(-ik_1)}\right]^{n_1} \left[\frac{\partial}{\partial(-ik_2)}\right]^{n_2} \cdots \left[\frac{\partial}{\partial(-ik_N)}\right]^{n_N} \tilde{p}(\vec{k}) \Bigg|_{\vec{k}=0}$$

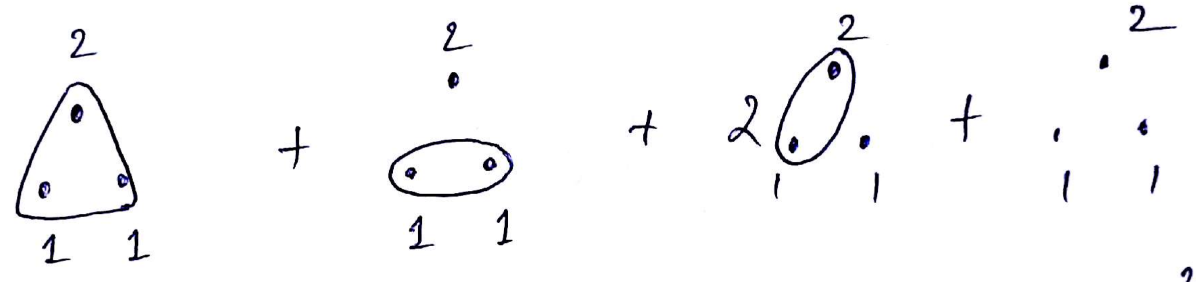
$$\langle x_1^{n_1} * x_2^{n_2} * \ldots x_N^{n_N} \rangle_c = \left[\frac{\partial}{\partial(-ik_1)}\right]^{n_1} \left[\frac{\partial}{\partial(-ik_2)}\right]^{n_2} \cdots \left[\frac{\partial}{\partial(-ik_N)}\right]^{n_N} \ln \tilde{p}(\vec{k}) \Bigg|_{\vec{k}=0}$$

The connected correlation $\langle x_\alpha * x_\beta \rangle_c$ is zero if $x_\alpha$ and $x_\beta$ are independent random variables.

## Graphical representation of moments:

$\langle x_1 x_2 \rangle =$ 

$\langle x_1 * x_2 \rangle_c \qquad \langle x_1 \rangle_c \langle x_2 \rangle_c$

$\langle x_1^2 x_2 \rangle =$ 

$\langle x_1^2 * x_2 \rangle_c \qquad \langle x_1^2 \rangle_c \langle x_2 \rangle_c \qquad 2 \langle x_1 * x_2 \rangle_c \langle x_1 \rangle_c \qquad \langle x_1 \rangle_c^2 \langle x_2 \rangle_c$