Central Limit Theorem:

Suppose x & y be two independent random variables.
$p(x)$ & $q(y)$ be their distributions.

What can we say about PDF of some $z = f(x,y)$? say $\phi(z)$

Lets try to construct some moments:

$$\langle z \rangle = \iint dx\, dy\, p(x)\, q(y)\, f(x,y)$$

$$\langle z^2 \rangle = \iint dx\, dy\, p(x)\, q(y) \left(f(x,y)\right)^2$$

$$\vdots$$

$$\langle z^n \rangle = \iint dx\, dy\, p(x)\, q(y) \left(f(x,y)\right)^n \quad \text{——— (1)}$$

This is as far as we can go!

There is another route.

Characteristic function or the generator of moments:

$$\tilde{\phi}(K) = \iint dx\, dy\, e^{-iK f(x,y)}\, p(x)\, q(y) \quad \text{——— (2)}$$

We can definitely proceed easily for a certain choice of

$z = f(x,y)$

Take $z = f(x,y) = x + y$ "sum of independent random variables"

Eq^n (2) then simplifies to : $\tilde{\phi}(K) = \tilde{p}(K)\, \tilde{q}(K)$ ——— (3)

"product of generating f^{n}s"

here $\tilde{p}(k) = \int e^{-ikx} p(x)\, dx$

& $\tilde{q}(k) = \int e^{-iky} q(y)\, dy$

A very important consequence of Eq^n (2) is the central limit theorem (discussed below):

Take a random variable x : $p(x)$ can be Gaussian, Exp., Cauchy etc.

Now form a variable $a = \frac{1}{N}\sum_{i=1}^{N} x_i$

Q: what is the distribution of a? i.e $P(a)$

A: $P(a)$ must be a Gaussian distribution.

Proof: We construct the generator of moments:

$$\tilde{P}(k) = \int..\int dx_1 \ldots dx_N\, p(x_1)..p(x_N)\, e^{-ika}$$

$$= \int..\int dx_1..dx_N\, p(x_1)..p(x_N)\, e^{-\frac{ik}{N}\sum_{i=1}^{N} x_i}$$

$$= \left(\tilde{p}(k/N)\right)^N \quad \text{where } \tilde{p}(k) = \int dx\, e^{-ikx} p(x)$$

(3)

Since $\tilde{p}(k) = \sum_{n=0}^{\infty} \frac{(-ik)^n \langle x^n \rangle}{n!}$

eq^n (3) becomes,

$$\tilde{P}(k) = \left(1 + \frac{(-ik)\langle x \rangle}{N} + \frac{(-ik)^2 \langle x^2 \rangle}{2N^2} + O[(k/N)^3]\right)^N$$

(4)

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Taking $N \to \infty$, eqn (4) becomes

$$\tilde{P}(k) = e^{-ik\langle x\rangle - k^2\langle x^2\rangle/(2N)} \qquad \text{—— (5)}$$

Taking inverse transform of above recovers the distrib

$$P(a) = \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-(a-\langle a\rangle)^2/(2\sigma^2)} \qquad \text{—— (6)}$$

Comparing (5) & (6), we find $\langle a\rangle = \langle x\rangle$ & $\sigma^2 = \langle x^2\rangle/N$

$$\therefore \quad P(a) = \frac{1}{(2\pi\langle x^2\rangle/N)^{1/2}} e^{-(a-\langle x\rangle)^2/(2\langle x^2\rangle/N)}$$

Conclusion: a is distributed normally with a mean equal to mean of x and variance $\langle x^2\rangle/N$

or

Sum of random numbers is normally distributed with mean equal to the mean of random numbers and a variance scaled by $1/N$ of the original.