$\Omega(E,V,N) \equiv$ Volume of available phase space.

The micro-state $\mu = \{\vec{q_i}, \vec{p_i}\}$ of an N-particle gas is just a single point in 6N-dimensional phase space.

If interactions are absent, we can write the Hamiltonian as

$$\mathcal{H} = \underbrace{\sum_{i=1}^{N} \frac{p_i^2}{2m}}_{\text{'Kinetic energy'}} + \underbrace{\sum_{i=1}^{N} u(\vec{q_i})}_{\text{'Box potential due to field'}}$$

The joint PDF in this micro-canonical ensemble is:

$$p(\mu) = \frac{1}{\Omega(E,V,N)}, \quad E-\Delta E \leq \sum_{i=1}^{N} \frac{p_i^2}{2m} \leq E+\Delta E \ \& \ \vec{q_i} \in \text{box}$$

$$= 0, \quad \text{otherwise}.$$

While particle positions could be anywhere inside the box, the allowed momenta must be within the hyper-spherical shell of thickness Δ_R.

where $\Delta_R = (2m)^{1/2} \left\{ (E+\Delta E)^{1/2} - (E-\Delta E)^{1/2} \right\}$

$\simeq (2m/E)^{1/2} \Delta E$, Taylor expanding $f(E) = \sqrt{E}$ to 1^{st} order. —— (1)

$$\left[f(E+\Delta E) = \sqrt{E+\Delta E} = (E)^{1/2} + \frac{\Delta E}{2\sqrt{E}} + O(\Delta E^2) \right]$$

The allowed phase space is thus

$$\Omega(E,V,N) = \int\cdots\int d^3\vec{q}_1 \cdots d^3\vec{q}_N \, d^3\vec{p}_1 \cdots d^3\vec{p}_N = V^N \Omega_p \quad \text{---} \quad (2)$$

where $\Omega_p = \int\cdots\int d^3\vec{p}_1 \cdots d^3\vec{p}_N$

Now Ω_p is the volume in momentum space of a hyper-spherical shell of thickness $\Delta_R = \sqrt{2m/E}\,\Delta E$.

We can thus write Ω_p as :

$$\Omega_p = \mathcal{A}_{3N}\,\Delta_R \quad \text{---} \quad (3)$$

where $\mathcal{A}_{3N} \equiv$ Area of the 3N-dimensional hyper sphere of radius $R = \sqrt{2mE}$

We write for $\mathcal{A}_{3N} = S_{3N} R^{3N-1}$ ——— (4)

where S_{3N} is the 3N-dimensional solid angle.

{ for eg. if $N = 1$, the 3-dimensional surface area becomes : $A_3 = S_3 R^2$ with $S_3 = 4\pi$ }

We will now calculate S_{3N} :

Consider the product of 3N Gaussian integrals.

$$I_{3N} = \left(\int_{-\infty}^{+\infty} dx\, e^{-x^2}\right)^{3N} = \pi^{3N/2} \quad \text{---} \quad (5) \qquad \because \int_{-\infty}^{+\infty} dx\, e^{-x^2} = \sqrt{\pi}$$

Alternatively, we may write

$$I_{3N} = \int \prod_{i=1}^{3N} dx_i \, e^{-x_i^2} = \int \prod_{i=1}^{3N} dx_i \, e^{-\sum_{j=1}^{3N} x_j^2}$$

Substituting for $\sum_{j=1}^{3N} x_j^2 = R^2$

and the volume element $\prod_{i=1}^{3N} dx_i = \underbrace{A_{3N}}_{\text{Area}} \; \underbrace{dR}_{\text{thickness}}$

$$\therefore \; I_{3N} = \int_0^\infty dR \, A_{3N} \, e^{-R^2} = \int_0^\infty dR \, S_{3N} \, R^{3N-1} e^{-R^2} \quad \ldots \text{from } (4)$$

change of variables : $R^2 = y \implies 2R\,dR = dy$

$$\therefore \; I_{3N} = \frac{S_{3N}}{2} \int_0^\infty dy \, y^{(3N/2 - 1)} e^{-y} = \frac{S_{3N}}{2} \left(\frac{3N}{2} - 1 \right)! \quad \text{—— } (6)$$

Comparing (5) & (6), we get:

$$S_{3N} = \frac{2 \pi^{3N/2}}{(3N/2 - 1)!} \quad \text{—— } (7)$$

Now combining (2), (3), (4), (5), (6) & (7), we get.

$$\Omega(E,V,N) = \frac{V^N \, 2 \pi^{3N/2} \, (2mE)^{(3N-1)/2}}{(3N/2 - 1)!} \, \Delta_R \quad \text{—— } (8)$$

The entropy is thus obtained as below

$$S(E,V,N) = k_B\left[N\ln V + \frac{3N}{2}\ln(2\pi m E) - \frac{3N}{2}\ln\left(\frac{3N}{2}\right) + \frac{3N}{2}\right]$$

$$= Nk_B \ln\left[V\left(\frac{4\pi e m E}{3N}\right)^{3/2}\right]$$

Properties of the ideal gas can be recovered from

$$Tds = dE + Pdv - \mu dN$$

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{N,V} = \frac{3}{2}\frac{Nk_B}{E}$$

$\therefore$ the internal energy $E = (3/2)Nk_B T$ is only a function of T.

The heat capacity is thus

$$C = \left(\frac{\partial E}{\partial T}\right) = \frac{3}{2}Nk_B$$

The equation of state is obtained from

$$\frac{P}{T} = \left(\frac{\partial S}{\partial V}\right)_{N,E} = \frac{Nk_B}{V}$$

$\therefore$ $PV = Nk_B T$ Ideal gas law.