GIAN COURSE

ORIGIN AND EVOLUTION OF PERTURBATIONS DURING INFLATION AND REHEATING

Department of Physics, Indian Institute of Technology Madras, Chennai

November 25–30, 2016

Lecture schedule

- The course will consist of eleven lectures and four tutorial sessions.
- The duration of each lecture and tutorial session will be 90 minutes. We will be meeting in MSB 360 for the lectures and in HSB 210 for the tutorial sessions.
- There will be two lectures on each of the first five days, and one on the last day. The lectures will be held during 08:45–10:15 AM and 10:30 AM–12:00 NOON. The first lecture will be on Friday, November 25, and the last lecture will be on Wednesday, November 30.
- The tutorial sessions will be held on Saturday, November 26, Sunday, November 27, Tuesday, November 29 and Wednesday, November 30. These sessions will be held during 2:00–3:30 PM on November 26, 27 and 29 and during 10:30 AM–12:00 NOON on November 30.

Information about the course

- A PDF file containing information about the course will be made available at the following URL:

  `https://www.physics.iitm.ac.in/~sriram/professional/events/gian-course/oepdir.pdf`

  We will keep updating this file as we make progress.

Syllabus and structure

Origin and Evolution of Perturbations during Inflation and Reheating

We have outlined below the syllabus for the course that we had originally proposed. The extent to which we will be able to cover the syllabus will depend on the background of the students and the speed at which progress is made during the course. At the end of the course, we will make available the syllabus that was eventually covered.

- **Lecture I: Background cosmology and the need for inflation**
  - Essential aspects of the Friedmann-Lemaître-Robertson-Walker universe
  - The hot big bang model – Drawbacks of the hot big bang model
  - The need for inflation – Duration of inflation – E-folds

- **Lecture II: Driving inflation with scalar fields**
  - The case of the canonical scalar field – Large and small field models
  - The potential and the Hubble slow roll parameters – Slow roll inflation – Hierarchy of slow roll parameters
  - k-inflation – Dirac-Born-Infeld model as a specific example

Exercise sheet I

- **Lectures III and IV: Linear cosmological perturbation theory**
  - Classification of perturbations – Decomposition theorem – Scalar, vector and tensor perturbations
  - Gauges and gauge invariant quantities
  - Scalar perturbations - Gauges – The first order Einstein and the stress-energy tensors – Equations of motion for perturbations induced by a hydrodynamical source
  - The case of multiple fluids – Adiabatic and iso-curvature perturbations – Conserved quantities
  - Vector perturbations – Equations of motion – Evolution
  - Tensor perturbations – Equations of motion – Evolution

Tutorial I

Exercise sheet II

Assignment I

- **Lectures V and VI: Generation of perturbations during inflation**
  - Equations of motion for a scalar field – The curvature perturbation
  - Quantization – The scalar and tensor power spectra in power law and slow roll inflation
  - Models of non-canonical scalar fields – Multiple scalar field models

Tutorial II

- **Lecture VII: Observational constraints on inflationary models**
  - The spectral indices and the consistency relation
  - Constraints from the CMB data on inflationary models

- The best inflationary models after Planck

- **Lecture VIII: The epoch of reheating and the evolution of perturbations**
  - Coarse grained reheating – The averaged equation of state
  - Equations of motion for interacting fields/fluids
  - First order, super-Hubble equations of motion for the curvature perturbations – Iso-curvature perturbations
  - The curvaton scenario

Exercise sheet III

Assignment II

- **Lecture IX and X: Generation of non-Gaussianities during inflation**
  - The Maldacena formalism – The third order actions governing the perturbations
  - The inflationary three-point functions
  - The scalar non-Gaussianity parameter
  - Non-Gaussianities generated in slow roll inflation
  - Observational constraints on the scalar non-Gaussianity parameter

Tutorial III

Assessment exam

Syllabus and structure

Origin and Evolution of Perturbations during Inflation and Reheating

We have outlined below the syllabus and structure that was eventually covered. Since no student had expressed interest in obtaining the credits offered for the course, there were no assignments or final assessment exam. We have included the problems that were originally planned for the assignments and assessment exam in the exercise sheets.

1. **Background cosmology and the need for inflation** [1.5 lectures]
   (a) Basics of general relativity
   (b) Friedmann-Lemaître-Robertson-Walker (FLRW) equations of motion
   (c) Particular solutions
   (d) The standard hot big bang model
   (e) The horizon problem
      i. Definition of a horizon
      ii. Recombination in brief
      iii. Angular size of the horizon
      iv. Angular size of the horizon today
      v. Solution: Inflation

Exercise sheet 1

2. **Driving inflation with scalar fields** [1.5 lectures]
   (a) The main idea
   (b) Equations of motion
   (c) The slow roll approximation
   (d) The slow roll trajectory
   (e) The scale factor during inflation
   (f) Example: Large field model

3. **The end of inflation (aka reheating)** [1 lecture]
   (a) Oscillations at the end of inflation
   (b) Inflaton's decay

Tutorial I

Exercise sheet II

4. **Linear cosmological perturbation theory** [2 lectures]
   (a) Classification of perturbations
   (b) Number of degrees of freedom
      i. Total degrees of freedom
      ii. Coordinate degrees of freedom
      iii. True degrees of freedom
   (c) Scalar perturbations
      i. Gauges

ii. Gauge transformations of metric perturbations
iii. Gauge transformation of the perturbed stress-energy tensor
iv. Gauge transformation of a scalar quantity

(d) Gauge invariant quantities
i. Bardeen variables (gauge invariant metric functions)
ii. Gauge invariant perturbations in the energy density, 'momentum' and pressure
iii. Gauge invariant scalar quantity

(e) Gauge invariant Einstein equations for the scalar perturbations
(f) Gauge invariant quantities for perturbations in scalar fields
(g) Tensor perturbations and the corresponding Einstein's equations

Tutorial 2
Exercise sheet 3

5. **Generation of perturbations during inflation** [2.5 lectures]

(a) Equations of motion for scalar perturbations
(b) Equations of motion for tensor perturbations (aka gravitational waves)
(c) Quantization
i. The Hamiltonian
ii. Schrodinger picture
iii. Initial Conditions

(d) Power spectrum
(e) Inflationary power spectrum in brief
(f) Inflationary power spectrum in the slow roll approximation
(g) Propagating the power spectrum in the post-inflationary universe

Exercise sheet 4

6. **Observational constraints on inflationary models** [1.5 lectures]

(a) Introduction
(b) Example: Large field model
(c) Calculating the number of e-folds between Hubble crossing and the end of inflation
(d) Predictions of large field inflation
(e) Planck and inflation

Tutorial 3

7. **Generation of non-Gaussianities during inflation** [1 lecture]

(a) The Maldacena formalism – The third order action governing the scalar perturbations
(b) The scalar bispectrum
(c) The scalar non-Gaussianity parameter
(d) Non-Gaussianities generated in slow roll inflation
(e) Observational constraints on the scalar non-Gaussianity parameter

Tutorial 4

Textbooks

1. S. Weinberg, *Gravitation and Cosmology* (John Wiley, New York, 1972).
2. E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley, New York, 1990).
3. S. Dodelson, *Modern Cosmology* (Academic Press, New York, 2003).
4. V. F. Mukhanov, *Physical Foundations of Cosmology* (Cambridge University Press, Cambridge, England, 2005).
5. S. Weinberg, *Cosmology* (Oxford University Press, Oxford, England, 2008).
6. R. Durrer, *The Cosmic Microwave Background* (Cambridge University Press, Cambridge, England, 2008).
7. D. H. Lyth and A. R. Liddle, *The Primordial Density Perturbation* (Cambridge University Press, Cambridge, England, 2009).
8. P. Peter, J-P. Uzan and J. Brujic, *Primordial Cosmology* (Oxford University Press, Oxford, England, 2009).

Older reviews

1. H. Kodama and M. Sasaki, *Cosmological perturbation theory*, Prog. Theor. Phys. Suppl. **78**, 1 (1984).
2. V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, *Theory of cosmological perturbations*, Phys. Rep. **215**, 203 (1992).
3. J. E. Lidsey, A. Liddle, E. W. Kolb, E. J. Copeland, T. Barreiro and M. Abney, *Reconstructing the inflation potential: An overview*, Rev. Mod. Phys. **69**, 373 (1997) [arXiv:astro-ph/9508078].
4. D. H. Lyth and A. Riotto, *Particle physics models of inflation and the cosmological density perturbation*, Phys. Rep. **314**, 1 (1999) [arXiv:hep-ph/9807278].
5. A. Riotto, *Inflation and the theory of cosmological perturbations*, arXiv:hep-ph/0210162.
6. J. Martin, *Inflation and precision cosmology*, Braz. J. Phys. **34**, 1307 (2004) [astro-ph/0312492].
7. J. Martin, *Inflationary cosmological perturbations of quantum-mechanical origin*, Lect. Notes Phys. **669**, 199 (2005) [arXiv:hep-th/0406011].
8. B. A. Bassett, S. Tsujikawa and D. Wands, *Inflation dynamics and reheating*, Rev. Mod. Phys. **78** 537 (2006).

Recent reviews

1. W. H. Kinney, *TASI lectures on inflation*, arXiv:0902.1529 [astro-ph.CO].
2. L. Sriramkumar, *An introduction to inflation and cosmological perturbation theory*, Curr. Sci. **97**, 868 (2009) [arXiv:0904.4584 [astro-ph.CO]].
3. D. Baumann, *Inflation*, arXiv:0907.5424 [hep-th].
4. J. Martin, *The observational status of cosmic inflation after Planck*, Astrophys. Space Sci. Proc. **45**, 41 (2016) [arXiv:1502.05733 [astro-ph.CO]].
5. H. Collins, *Primordial non-Gaussianities from inflation*, arXiv:1101.1308 [astro-ph.CO].

Exercise sheet 1

Background cosmology and the need for inflation

1. *Planck mass:* The so-called reduced Planck mass $m_{\rm Pl}$ is a fundamental quantity, evidently, with the dimensions of mass, constructed out of the constants G, $\hbar$ and c.

   (a) Show that, in $\hbar = c = 1$ units, $m_{\rm Pl} = 1/\sqrt{G}$.

   (b) A quantity which we will repeatedly encounter in the Einstein's equations is $8\,\pi\,G$ and we shall define $M_{\rm Pl} = 1/\sqrt{8\,\pi\,G}$. Show that $M_{\rm Pl} \simeq (1.22/\sqrt{8\,\pi}) \times 10^{19}\,{\rm GeV}$.

2. *Hubble constant and critical density:* The Hubble constant has been observationally determined to be $H_0 \simeq 70$ km s^{-1} Mpc^{-1}.

   (a) Express H_0 in units of GeV.

   (b) Recall that the critical density of the universe today is given by $\rho_{\rm C} = 3\,H_0^2/(8\,\pi\,G)$. Given the above value of H_0, determine $\rho_{\rm C}$ and also express it in units of GeV^4.

   Note: $1\,{\rm Mpc} = 3.26 \times 10^6$ light years.

3. *Spatial curvature of the Friedmann metric:* Recall that the Friedmann metric is described by the line-element

$$
{\rm d}s^2 = -{\rm d}t^2 + a^2(t)\,\left[\frac{{\rm d}r^2}{1-k\,r^2} + r^2\,\left({\rm d}\theta^2 + \sin^2\theta\,{\rm d}\phi^2\right)\right].
$$

   Show that the *spatial* curvature of this line-element is given by ${}^3R = 6\,k/a^2$.

4. *Behavior of momentum and energy in a Friedmann universe:* Consider a material particle propagating along a geodesic in a Friedmann universe.

   (a) Let the particle be traveling along $\theta =$ constant and $\phi =$ constant. Show that the zeroth component of the geodesic equation reads as

$$
\frac{{\rm d}^2t}{{\rm d}s^2} + \left(\frac{a\,\dot{a}}{1-k\,r^2}\right)\left(\frac{{\rm d}r}{{\rm d}s}\right)^2 = 0.
$$

   (b) Eliminate $({\rm d}r/{\rm d}s)$ between this equation and the first integral (*viz.* $u^\mu\,u_\mu = -1$)

$$
\left(\frac{{\rm d}t}{{\rm d}s}\right)^2 - \left(\frac{a^2}{1-k\,r^2}\right)\left(\frac{{\rm d}r}{{\rm d}s}\right)^2 = 1
$$

   to obtain that

$$
\left(\frac{{\rm d}^2t}{{\rm d}s^2}\right) + \frac{\dot{a}}{a}\,\left[\left(\frac{{\rm d}t}{{\rm d}s}\right)^2 - 1\right] = 0.
$$

   (c) Integrate this equation to arrive at

$$
a^2\,\left[\left(\frac{{\rm d}t}{{\rm d}s}\right)^2 - 1\right] = {\rm constant}.
$$

   (d) Using this result and the constraint $u^\mu\,u_\mu = -1$, show that

$$
\sigma_{ij}\,p^i\,p^j = \vert{\bf p}\vert^2 \propto a^{-2},
$$

   where σ_{ij} represents the spatial part of the Friedmann metric.

   (e) Repeat the exercise for a photon and show that, in a Friedmann universe, the frequency $\omega(t)$ of the photon behaves as $\omega(t) \propto 1/a(t)$.

5. *Behavior of temperature in a Friedmann universe:* The number density of particles within the phase space volume $\mathrm{d}^3\boldsymbol{x}\,\mathrm{d}^3\boldsymbol{p}$ is given by

$$\mathrm{d}N = f(\boldsymbol{x},\boldsymbol{p},t)\,\mathrm{d}^3\boldsymbol{x}\,\mathrm{d}^3\boldsymbol{p},$$

where $f(\boldsymbol{x},\boldsymbol{p},t)$ denotes the distribution function. In a Friedmann universe, the distribution function will be independent of $\boldsymbol{x}$ due to the homogeneity of the background, and it will depend only on $|\boldsymbol{p}|$ (rather than on $\boldsymbol{p}$) due to the isotropy.

   (a) Show that, if no particles are created or destroyed, then the distribution function remains invariant during the evolution of the universe.

   (b) Recall that the distribution function describing a homogeneous collection of photons at a finite temperature is given by

$$f_\gamma(\boldsymbol{k}) = \frac{g}{(2\,\pi)^3}\,\frac{1}{\exp\left(\omega/T\right)-1},$$

   where $\omega = |\boldsymbol{k}|$ is the frequency of the photon, g is its spin-degeneracy (*viz.* 2) and T is the temperature. Argue that, for such a distribution of photons in a Friedmann universe, the invariance of the distribution function implies that the temperature of the radiation is inversely proportional to the scale factor.

6. *Equation of state for radiation:* The energy density and pressure associated with a thermal distribution of photons are given by

$$\rho = \int \mathrm{d}^3\boldsymbol{k}\, f_\gamma(\mathbf{k})\,\omega, \qquad p = \int \mathrm{d}^3\boldsymbol{k}\, f_\gamma(\boldsymbol{k})\,\left(\frac{k^2}{3\,\omega}\right),$$

where $f_\gamma(\boldsymbol{k})$ is the distribution function given in the previous exercise. Without having to explicitly evaluate the integrals, show that $p = \rho/3$ for a thermal distribution of photons.

7. *Epoch of equality:* The Cosmic Microwave Background (CMB) is considered to be the dominant contribution to the relativistic energy density in the universe. The second most dominant contribution arises due to the three species nearly massless neutrinos.

   (a) Given that the temperature of the CMB today is $T_\gamma \simeq 2.725\,\mathrm{K}$, show that

$$\Omega_\gamma\,h^2 \simeq 2.443\times 10^{-5},$$

   where h is related to the Hubble constant H_0 as follows:

$$H_0 \simeq 100\;h\;\mathrm{km\,s^{-1}\,Mpc^{-1}}.$$

   (b) The temperature of neutrinos today is expected to be $T_\nu = (4/11)^{1/3}\,T_\gamma$. Given that there are three species of neutrinos and their spin degeneracy is two, show that

$$\Omega_\nu\,h^2 \simeq 0.681\,\Omega_\gamma\,h^2.$$

   (c) Show that the redshift z_{eq} at which the energy density of matter and radiation (including photons as well as neutrinos) were equal is given by

$$1+z_{\mathrm{eq}} = \frac{\Omega_{\mathrm{NR}}}{\Omega_{\mathrm{R}}} \simeq 2.435\times 10^4\,\left(\Omega_{\mathrm{NR}}\,h^2\right).$$

   (d) If $\Omega_{\mathrm{NR}} = \Omega_{\mathrm{CDM}} + \Omega_{\mathrm{B}}$ and

$$\Omega_{\mathrm{CDM}}\,h^2 = 0.12, \qquad \Omega_{\mathrm{B}}\,h^2 = 0.022,$$

   show that $z_{\mathrm{eq}} \simeq 3457$.

8. *Cosmological parameters and the age of the universe:* Recall that the scale factor is related to the redshift z as

$$1 + z = \frac{a_0}{a}.$$

   (a) Rewrite the first of the Friedmann equations in terms of the cosmological parameters as follows:

$$\left(\frac{H(z)}{H_0}\right)^2 = \Omega_{\rm NR}\,(1+z)^3 + \Omega_{\rm R}\,(1+z)^4 + \Omega_\Lambda - (\Omega - 1)\,(1+z)^2,$$

   where $\Omega = \Omega_{\rm NR} + \Omega_{\rm R} + \Omega_\Lambda$.

   (b) Using this expression, express the age of the universe in terms of the cosmological parameters $\Omega_{\rm NR}$, $\Omega_{\rm R}$, Ω_Λ and H_0.

   (c) Estimate the age of the spatially flat, matter dominated universe given that the Hubble parameter today is $H_0 \simeq 70\ {\rm km\,s^{-1}\,Mpc^{-1}}$.

9. *Piecewise solutions to the scale factor:* Consider a brief interval (say, over $t_{\rm i} < t < t_{\rm e}$) during the early stages of the radiation dominated epoch wherein the universe was dominated by a fluid with an equation of state parameter w. Let the universe later transit to the matter dominated at $t_{\rm eq}$. Matching the scale factor and its time derivative at $t_{\rm i}$, $t_{\rm e}$ and $t_{\rm eq}$, show that the scale factor during the different epochs can be expressed as

$$a(t) = \begin{cases} a_{\rm i}\,(2\,H_{\rm i}\,t)^{1/2}, & 0 < t < t_{\rm i}, \\ a_{\rm i}\,\left[\frac{3}{2}\,(1+w)\,H_{\rm i}\,(t - t_{\rm i}) + 1\right]^{2/[3\,(1+w)]}, & t_{\rm i} < t < t_{\rm e}, \\ a_{\rm e}\,\left[2\,H_{\rm e}\,(t - t_{\rm e}) + 1\right]^{1/2}, & t_{\rm e} < t < t_{\rm eq}, \\ a_{\rm eq}\,\left[\frac{3}{2}\,H_{\rm eq}\,(t - t_{\rm eq}) + 1\right]^{2/3}, & t_{\rm eq} < t < t_0, \end{cases}$$

where $(a_{\rm i}, H_{\rm i})$, $(a_{\rm e}, H_{\rm e})$ and $(a_{\rm eq}, H_{\rm eq})$ denote the scale factor and the Hubble parameter at the times $t_{\rm i}$, $t_{\rm e}$ and $t_{\rm eq}$, respectively.

10. *Solving the Friedmann equations with two components:* We had earlier discussed the solutions to the Friedmann equations during an epoch dominated by a single component of matter. It proves to be difficult to analytically solve the Friedmann equations when all three (*viz.* matter, radiation and the cosmological constant) are present. However, one finds that solutions can be obtained in the presence of two components.

   (a) Integrate the Friedmann equation for a $k = 0$ universe with matter and radiation to obtain that

$$a(\eta) = \sqrt{a_0\,\Omega_{\rm R}}\,(H_0\,\eta) + \left(\frac{\Omega_{\rm NR}\,a_0^2}{4}\right)(H_0\,\eta)^2,$$

   where η is the conformal time coordinate.

   Note: In obtaining the above result, it has been assumed that $a = 0$ at $\eta = 0$.

   (b) Integrate the Friedmann equation for a $k = 0$ universe with matter and cosmological constant to obtain that

$$\frac{a}{a_0} = \left(\frac{\Omega_{\rm NR}}{\Omega_\Lambda}\right)^{1/3}\sinh^{2/3}\left(\frac{3\,\sqrt{\Omega_\Lambda}\,H_0\,t}{2}\right),$$

   where we have set the constant of integration to be zero.

   (c) Discuss the large and small time behavior of these solutions and show that they reduce to the required limit.

Exercise sheet 2

Driving inflation with scalar fields

1. *Scalar fields in curved spacetime:* Recall that the action describing a scalar field, say, ϕ, in a curved spacetime is given by

$$\mathcal{S}[\phi] = -\int \mathrm{d}^4x\,\sqrt{-g}\,\left[\frac{1}{2}\,g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi + V(\phi)\right],$$

where $V(\phi)$ is the potential describing the scalar field.

   (a) Vary the above action with respect to the scalar field to arrive at the following equation of motion governing the field:

$$\Box\phi - V_\phi = \frac{1}{\sqrt{-g}}\,\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\,\partial_\nu\right)\phi - V_\phi = 0,$$

   where $V_\phi = \mathrm{d}V/\mathrm{d}\phi$.

   (b) Show that in a flat, Friedmann model described by the line-element

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\,\mathrm{d}\boldsymbol{x}^2 = a^2(\eta)\,\left(-\mathrm{d}\eta^2 + \mathrm{d}\boldsymbol{x}^2\right),$$

   a homogeneous scalar field satisfies the equation

$$\ddot{\phi} + 3\,H\,\dot{\phi} + V_\phi = \phi'' + 2\,\mathcal{H}\,\phi' + V_\phi\,a^2 = 0,$$

   where H is the Hubble parameter, while $\mathcal{H} = (a'/a)$ denotes the conformal Hubble parameter, and the dots and primes (here and hereafter) denote differentiation with respect to the cosmic time t and the conformal time η.

   (c) Vary the above action with respect to the metric tensor to arrive at the following express for the stress-energy tensor of the scalar field

$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\,\left[\frac{1}{2}\,g^{\alpha\beta}\,\partial_\alpha\beta\,\partial_\nu\phi + V(\phi)\right].$$

   (d) Show that, for a homogeneous scalar field in a Friedmann universe

$$T^t_t = -\rho = -\frac{\dot{\phi}^2}{2} - V(\phi), \qquad T^i_j = p\,\delta^i_j = \left[\frac{\dot{\phi}^2}{2} - V(\phi)\right]\,\delta^i_j.$$

   (e) Show that the conservation of the stress-energy tensor also leads to the above equation of motion for the scalar field.

2. *Reverse engineering a potential:* The Friedmann equations for a flat, scalar field dominated universe can be written as

$$H^2 = \frac{1}{3\,M_{_{\mathrm{Pl}}}^2}\,\left[\frac{\dot{\phi}^2}{2} + V(\phi)\right], \qquad \frac{\ddot{a}}{a} = -\frac{1}{3\,M_{_{\mathrm{Pl}}}^2}\,\left[\dot{\phi}^2 - V(\phi)\right].$$

Using these Friedmann equations, show that the scalar field and the potential can be expressed parametrically in terms of the coordinate time t as follows:

$$\phi(t) = \sqrt{2}\,M_{_{\mathrm{Pl}}}\int \mathrm{d}t\,\left(H^2 - \frac{\ddot{a}}{a}\right)^{1/2}, \qquad V(t) = M_{_{\mathrm{Pl}}}^2\,\left(2\,H^2 + \frac{\ddot{a}}{a}\right).$$

Note: Given the scale factor $a(t)$, these two equations allows us to construct the potential from which such a scale factor can arise.

3. <u>*Exact solutions:*</u> The above results allow us to construct some exact solutions to the Friedmann equations and the equation of motion governing the scalar field.

   (a) Show that the potential

$$V(\phi) = V_0 \exp\left(-\sqrt{\frac{2}{p}}\,\frac{\phi}{M_{\rm Pl}}\right),$$

   where V_0 and p are constants, leads to the following behavior for $a(t)$ and $\phi(t)$

$$a(t) = a_0\, t^p, \qquad \frac{\phi(t)}{M_{\rm Pl}} = \sqrt{2\,p}\ \ln\left\{\left[\frac{V_0}{(3\,p-1)\,p}\right]^{1/2} \frac{t}{M_{\rm Pl}}\right\}.$$

   (b) Show that the potential

$$V(\phi) = 3\,\alpha^2\,\beta^2\,\gamma^\kappa\,M_{_{\rm Pl}}^2\,\left(1 - \frac{\kappa^2\,M_{\rm Pl}^2}{6\,\phi^2}\right)\,\left(\frac{\phi}{M_{\rm Pl}}\right)^{-\kappa},$$

   where $\gamma = \sqrt{2\,\alpha\,\kappa}$ and $\kappa = 4\,(1-\beta)/\beta$, leads to the following behavior for $a(t)$

$$a(t) = a_0\,\exp\left(\alpha\,t^\beta\right), \quad \alpha > 0,\ 0 < \beta < 1.$$

4. <u>*The first slow roll parameter:*</u> Show that we can write the first slow roll parameter as

$$\epsilon_1 = -\frac{\dot{H}}{H^2} = 1 - \frac{\mathcal{H}'}{\mathcal{H}^2},$$

   where $\mathcal{H} = a'/a$ is the conformal Hubble parameter.

5. <u>*The second slow roll parameter in the slow roll approximation:*</u> Show that, in the slow roll approximation, the second slow roll parameter ϵ_2 is given by

$$\epsilon_2 \simeq 2\,M_{_{\rm Pl}}^2\,\left[\left(\frac{V_\phi}{V}\right)^2 - \frac{V_{\phi\phi}}{V}\right].$$

   In fact, the higher order slow roll parameters will involve suitably higher derivatives of the potential. Show that, for instance, the third slow roll parameter ϵ_3 depends on the third derivative of the potential.

6. <u>*Solutions in the slow roll approximation:*</u> As it was done in the case of large field models, obtain the solutions to the scalar field for small field models described by potentials of the form

$$V(\phi) = M^4\,\left[1 - \left(\frac{\phi}{\mu}\right)^p\right].$$

7. <u>*k-inflation:*</u> The action for a generic scalar field can be written as

$$\mathcal{S}[\phi] = \int {\rm d}^4x\,\sqrt{-g}\,\mathcal{L}(X,\phi),$$

   where

$$X = -\frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi.$$

   (a) Show that, in a spatially flat Friedmann universe, the energy density ρ and pressure p associated with such a scalar field are given by

$$\rho = -2\,X\,\left(\frac{\partial\mathcal{L}}{\partial X}\right) + \mathcal{L}, \qquad p = \mathcal{L}.$$

(b) Show that the scalar field ϕ described by the above action satisfies the following equation of motion:

$$\left[\frac{\partial \mathcal{L}}{\partial X} + 2\,X\,\left(\frac{\partial^2 \mathcal{L}}{\partial X^2}\right)\right]\ddot{\phi} + \left[3\,H\,\left(\frac{\partial \mathcal{L}}{\partial X}\right) + \dot{\phi}\,\left(\frac{\partial^2 \mathcal{L}}{\partial X\,\partial \phi}\right)\right]\dot{\phi} - \left(\frac{\partial \mathcal{L}}{\partial \phi}\right) = 0.$$

(c) Verify that this equation of motion for the scalar field ϕ is consistent with the conservation equation

$$\dot{\rho} = -3\,H\,(\rho + p).$$

8. *Tachyons:* Tachyons T are defined by an action of the following form:

$$\mathcal{S}[T] = -\int \mathrm{d}^4x\,\sqrt{-g}\,V(T)\,\sqrt{1 - \partial_\mu T\,\partial^\mu T}.$$

(a) Derive the stress-energy tensor corresponding to this action.

(b) Construct the energy density ρ and the pressure p associated with the tachyon in a Friedmann universe.

(c) Obtain the equation governing the tachyon.

(d) For the potential

$$V(T) = \frac{\lambda}{\cosh\,(T/T_0)},$$

identify the condition or domain over which inflation can occur.

9. *Evolution of the scalar fields during preheating:* Consider the evolution of the scalar field in a quadratic potential after inflation has terminated. We had seen earlier that, in such a case, the evolution mimics that of a matter dominated epoch. Solving the equation of motion of the scalar field under these conditions, show that the scalar field behaves as

$$\phi(t) \simeq \phi_{\mathrm{end}}\,\sin\,(m\,t),$$

where ϕ_{end} is the value of the scalar field at the end of inflation.

10. *The reheating temperature:* We had discussed that, when the scalar field is coupled to radiation near the minima of a large field model of the form $V = M^4\,(\phi/M_{_{\mathrm{Pl}}})^n$, the radiation energy density behaves as

$$\rho_\gamma \simeq \rho_{\mathrm{end}}\,\left(\frac{a}{a_{\mathrm{end}}}\right)^{-4}\,\left(\frac{n+2}{2\,n\,\Gamma\,t_{\mathrm{end}}}\right)^{(8-2\,n)/(3\,n)}\,\left[\gamma\left(\frac{n+8}{3\,n}, \frac{2\,n\,\Gamma\,t}{n+2}\right) - \gamma\left(\frac{n+8}{3\,n}, \frac{2\,n\,\Gamma\,t_{\mathrm{end}}}{n+2}\right)\right],$$

where Γ is the decay constant and $\gamma(\alpha, x)$ denotes the incomplete Gamma function.

(a) For small x, $\gamma(\alpha, x)$ behaves as

$$\gamma(\alpha, x) \simeq \frac{x^\alpha}{\alpha}.$$

Using this result, show that we can write

$$\rho_\gamma \simeq \Gamma\,\rho_{\mathrm{end}}\,t_{\mathrm{end}}^2\,\left[\frac{6\,n^2}{(n+2)\,(n+8)\,t}\right]\,\left[1 - \left(\frac{t}{t_{\mathrm{end}}}\right)^{-(n+8)/(3\,n)}\right],$$

which is valid for $t_{\mathrm{end}} \le t \le t_{_{\mathrm{RH}}}$.

(b) Assuming $t_{_{\mathrm{RH}}} \gg t_{\mathrm{end}}$ and pushing the limits of the approximation involved in arriving at the above expression, show that

$$\rho_\gamma \simeq \Gamma\,\rho_{\mathrm{end}}\,t_{\mathrm{end}}^2\,\left[\frac{6\,n^2}{(n+2)\,(n+8)\,t_{_{\mathrm{RH}}}}\right].$$

(c) Equating this to

$$\rho_\gamma = \frac{g_* \, \pi^2 \, T^4}{30},$$

arrive at the result that the reheating temperature $T_{\rm RH}$ can be expressed as

$$T_{\rm RH} \propto (\Gamma \, M_{\rm Pl})^{1/2} \, .$$

Exercise sheet 3

Evolution of perturbations

1. <u>*Need for accelerated expansion:*</u> Show that, if the modes should be inside the Hubble radius as *one goes back in time*, then the universe needs to go through an accelerated phase of expansion.

2. <u>*Initial conditions:*</u> Assuming power law expansion, show that, during non-accelerating phases the horizon and the Hubble radius are proportional to each other. Calculate the corresponding quantities during an inflationary epoch assuming, say, exponential inflation. What do you find?

3. <u>*Bardeen equation:*</u> Assuming that no anisotropic stresses are present, combine the first order Einstein's equations to arrive at the following equation governing Bardeen potential:
$$\Phi'' + 3\,\mathcal{H}\,\left(1+c_{\mathrm{A}}^2\right)\,\Phi' - c_{\mathrm{A}}^2\,\nabla^2\Phi + \left[2\,\mathcal{H}' + \left(1+3\,c_{\mathrm{A}}^2\right)\,\mathcal{H}^2\right]\,\Phi = 4\,\pi\,G\,a^2\,\delta p^{\mathrm{NA}},$$
where $\mathcal{H} = a\,H$ is the conformal Hubble parameter and δp^{NA} is the non-adiabatic pressure perturbation defined as
$$\delta p^{\mathrm{NA}} = \delta\mathcal{P} - c_{\mathrm{A}}^2\,\delta\varrho = \delta p - c_{\mathrm{A}}^2\,\delta\rho,$$
with $\delta\varrho$ and $\delta\mathcal{P}$ denoting the gauge invariant density and pressure perturbations, while $c_{\mathrm{A}}^2 = \dot{p}/\dot{\rho}$ represents the adiabatic speed of the perturbations.

4. <u>*Conservation of the curvature perturbation:*</u> Consider the following linear combination of the Bardeen potential Ψ and the gauge invariant energy flux $\delta\varsigma$:
$$\mathcal{R} = \Psi + \left(\frac{H}{\rho+p}\right)\,\delta\varsigma,$$
a quantity that is referred to as the curvature perturbation.
   (a) In the absence of anisotropic stresses, upon using the first order Einstein's equations, show that the quantity $\mathcal{R}$ defined above can be expressed as
   $$\mathcal{R} = \Phi + \frac{2\,\rho}{3\,\mathcal{H}}\,\left(\frac{\Phi' + \mathcal{H}\,\Phi}{\rho+p}\right).$$
   (b) Upon substituting this expression in the above equation that describes the evolution of the potential Φ, and making use of the background equations, show that one obtains that, in Fourier space,
   $$\mathcal{R}_k' = \frac{\mathcal{H}}{\mathcal{H}^2 - \mathcal{H}'}\,\left(4\,\pi\,G\,a^2\,\delta p_k^{\mathrm{NA}} - c_{\mathrm{A}}^2\,k^2\,\Phi_k\right).$$
   (c) Using this equation, argue that, when the non-adiabatic pressure perturbations are absent, the curvature perturbation is conserved on super Hubble scales.

5. <u>*Evolution of the Bardeen potential:*</u> Consider the equations above that govern the evolution of the Bardeen potential and the curvature perturbation.
   (a) Argue that when the equation of state parameter w is a constant, $c_{\mathrm{A}}^2 = w$.
   (b) Show that, in a matter dominated epoch, the Bardeen potential is a constant on super Hubble *as well as* sub Hubble scales.
   (c) Using the conservation of the curvature perturbation, show that the Bardeen potential at late times during the matter dominated epoch, say, Φ, is related to the Bardeen potential at early times during the radiation dominated epoch, say, Φ_{P}, as $\Phi \simeq 9\,\Phi_{\mathrm{P}}/10$, for modes that remain on super Hubble scales.
   (d) For the scale factor that we had obtained earlier when matter as well as radiation are present, it is possible to solve the above Bardeen equation completely analytically for super Hubble modes. Attempt to obtain the solution and, using the solution, illustrate explicitly that $\Phi \simeq 9\,\Phi_{\mathrm{P}}/10$.

Exercise sheet 4

Generation of perturbations during inflation

1. *Scalar action in Fourier space:* Given the action describing the Mukhanov-Sasaki variable in real space, obtain the following action governing the scalar perturbations in Fourier space:

$$\begin{aligned} {}^2\delta\mathcal{S} &= \int \mathrm{d}\eta \int \mathrm{d}^3\boldsymbol{k}\, \frac{1}{2}\left[v_{\boldsymbol{k}}{}'\, v_{\boldsymbol{k}}^{*}{}' - \left(k^2 - \frac{z''}{z}\right) v_{\boldsymbol{k}}\, v_{\boldsymbol{k}}^{*}\right] \\ &= \int \mathrm{d}\eta \int_{\boldsymbol{k}>0} \mathrm{d}^3\boldsymbol{k}\, \frac{1}{2}\left[v_{\boldsymbol{k}}{}'\, v_{\boldsymbol{k}}^{*}{}' - \left(k^2 - \frac{z''}{z}\right) v_{\boldsymbol{k}}\, v_{\boldsymbol{k}}^{*}\right] \end{aligned}$$

2. *z''/z in terms of the slow roll parameters:* Show that, in the slow roll approximation, one can write,

$$\frac{z''}{z} = -\frac{1}{\eta^2}\left(2 + 3\,\epsilon_1 - \frac{3}{2}\,\epsilon_2\right).$$

3. *Gravitational waves in de Sitter:* Consider the tensor modes in de Sitter spacetime.
   (a) Working in terms of the conformal time coordinate, solve the equation governing the tensor modes in de Sitter spacetime.
   (b) Impose the Bunch-Davies initial conditions in the sub Hubble limit to identify the positive frequency normal modes.
   (c) Show that the amplitude of the tensor modes freeze on super Hubble scales.
   (d) Evaluate the tensor power spectrum in the super Hubble limit.

4. *Conservation of the perturbed energy density:* Suitably perturb the equation governing the stress-energy tensor and arrive at the equation describing the conservation of the perturbed energy density.

   Note: Recall that this equation was required to show that ζ_{BST} was conserved on super Hubble scales.