Black holes and their QNMs in degenerate EGB Gravity

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Plan of my talk

- What is Einstein Gauss Bonnet
- Why degenerate metrics
- QNMs
- How are they all related ??

Origin of Gauss-Bonnet in gravity??

- The action for Lovelock gravity in D dimensions is given by(Lovelock 1972):

$$S = \int d^D x \sqrt{-g_D} \sum_{p=0}^{\frac{(D-1)}{2}} \alpha_p \mathcal{L}_p$$

- where α_p is a coupling constant and $\mathcal{L}_p := \frac{1}{2^p} \delta^{A_1 B_1 \ldots A_p B_p}_{C_1 D_1 \ldots C_p D_p} \mathcal{R}^{C_1 D_1}_{A_1 B_1} \ldots \mathcal{R}^{C_p D_p}_{A_p B_p}$
- $\mathcal{L}_0 = 1$, $\mathcal{L}_1 = \mathcal{R}$, $\mathcal{L}_2 \equiv \mathcal{G} = \mathcal{R}^2 - 4\mathcal{R}^{AB}\mathcal{R}_{AB} + \mathcal{R}^{ABCD}\mathcal{R}_{ABCD}$
- In four dimensions the Euler characteristic is defined as an integral over the Gauss-Bonnet term.
- In four dimensions Gauss-Bonnet density is topological and does not contribute to dynamics.

How to construct theories with Gauss-Bonnet density??

- Theories in higher dimensions than $(3+1)$: Usually string inspired
- Scalar tensor theories of the hondeski class
- Kaluza-Klein reduction of a higher $(D > 4)$ dimensional Einstein Gauss-Bonnet action
- A rescaling the coupling constant $\alpha_2 \to \frac{\alpha_2}{(D-4)}$
- Issues with these methods
- The addition scalar degrees of freedom in scalar tensor theories do not decouple form the emergent metric.
- The $D \to 4$ limit is not valid unless Bianchi identities are violated(Metin Gürses 2020)
- Possible to construct a framework without such issues ?? $\to$ **degenerate EGB (S.Sengupta 2022)**

What are degenerate metrics?

- The theory of gravitational interactions is described by the framework of general relativity.
- The independent field in this framework is the metric tensor $g_{\mu\nu}$.
- The Einstein Hilbert action is given by:

$$S = \int d^4x \sqrt{-g} g^{\mu\nu} g^{\alpha\beta} R_{\alpha\mu\beta\nu}$$

- Equations of motion:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$$

- These equations **do not admit** non-invertible metrics as solutions.

- Can we have a framework which admits both invertible and non-invertible metrics???
- Another formulation of gravity based on $SO(3,1)$ gauge fields of gravity.
- The basic field: tetrad e_μ^I and spin connections ω_μ^{IJ}.
- Action:

$$S = \int d^4x \epsilon^{IJKL} \epsilon^{\mu\nu\alpha\beta} e_{\mu I} e_{\nu J} R_{\alpha\beta KL}(\omega)$$

- Variation with respect to the independent fields e_μ^I and ω_μ^{IJ}:

$$\epsilon^{\mu\nu\alpha\beta}\epsilon_{IJKL} e_\nu^J R_{\alpha\beta}^{KL} = 0,$$
$$\epsilon^{\mu\nu\alpha\beta}\epsilon_{IJKL} e_\nu^K D_\alpha(\omega) e_\beta^L = 0$$

Equations of motion

$$\epsilon^{\mu\nu\alpha\beta}\epsilon_{IJKL}e^{J}_{\nu}R^{KL}_{\alpha\beta}=0,$$
$$\epsilon^{\mu\nu\alpha\beta}\epsilon_{IJKL}e^{K}_{\nu}D_{\alpha}(\omega)e^{L}_{\beta}=0$$

- For the invertible case these equations reduce to standard Einstein field equations.
- Unlike the metric formulation the first order formulation is **well defined** for non-invertible tetrads.
- For the non-invertible case(Kaul-Sengupta 2016,2017):
- 24 connection fields ω_a^{ij}; 12 get fixed, rest undetermined.
- Further constraints from second set of equations of motions.
- Generically contain torsion even in the absence of any matter fields.
- Equations of motion also admit vacuum solutions- can be degenerate everywhere or over a region or at isolated points (Tseytlin.1982, Kaul-Sengupta 2016,2017).

Why are these solutions important??

- A priori their is no reason to ignore these solutions!
- Torsion in the absence of any matter fields.
- Possible resolution to long standing problems, Eg: curvature singularities(Kaul-Sengputa 2017) , causal singularities(open issue) ,etc.
- A purely geometric interpretation of magnetic monopoles
- Interpretation of degenerate metric geometry via regularised point electric charges.
- Zero determinant phase be used to construct a four or lower dimensional theory of gravity.

A Detour to 2-D gravity

- The Einstein Hilbert action in two dimension is topological
- No local gravitational dynamics
- **How to construct a possible dynamical theory ??**
- **Mann & Ross 1993, R. Jackiw 1984, C. Teitelboim 1984...**

Objective:

To define an unique metric theory in two dimensions from the perspective of gravity in presence of extra dimensions of vanishing proper length.

Gravity theory with degenerate direction

- Conside the Lagrangian:

$$\mathcal{L}(\hat{e},\hat{\omega}) = \epsilon^{\mu\nu\alpha}\epsilon_{IJK}\left[\xi \hat{e}^I_\mu \hat{R}_{\nu\alpha}{}^{JK}(\hat{\omega}) + \frac{\beta}{3}\hat{e}^I_\mu \hat{e}^J_\nu \hat{e}^K_\alpha\right]$$

- Triad $\hat{e}^I_\mu$ and spin connection $\hat{\omega}_\mu{}^{IJ}$ are the independent fields.
- ξ Gravitational coupling and β cosmological constant.
- Taking the variation

$$\epsilon^{\mu\nu\alpha}\epsilon_{IJK}\hat{D}_\mu(\hat{\omega})\hat{e}^I_\nu = 0$$

$$\epsilon^{\mu\nu\alpha}\epsilon_{IJK}\left[\xi \hat{R}_{\mu\nu}{}^{IJ}(\hat{\omega}) + \beta \hat{e}^I_\mu \hat{e}^J_\nu\right] = 0$$

- We are interested in the solution space corresponding to triad fields with one vanishing eigen-value.
- Assuming the degenerate direction to be v i.e $g_{v\mu} = 0$

$$\hat{e}^I_\mu = \begin{bmatrix} \hat{e}^i_a \equiv e^i_a & \hat{e}^2_a = 0 \\ \hat{e}^i_v = 0 & \hat{e}^2_v = 0 \end{bmatrix}$$

- $\mu \equiv (t, x, v) \equiv (a, v)$ & $I \equiv (0, 1, 2) \equiv (i, 2)$
- The diad field e^i_a along with torsionless spin connection $\bar{\omega}_\mu{}^{IJ}$ define the emergent two-dimensional gravity.
- The emergent anti symmetric densities are: $\epsilon^{vab} \equiv \epsilon^{ab}$ and $\epsilon_{2ij} \equiv \epsilon_{ij}$

- Solving the first set of equations :

$$\begin{aligned}
\alpha &= v,\ (J,K)=(j,2): \quad \epsilon^{ab}\epsilon_{ij}\hat{D}_a\hat{e}^i_b = 0 = \hat{D}_{[a}\,\hat{e}^i_{\ b]} \implies K_a{}^{ij} \equiv \hat{\omega}_a{}^{ij} - \bar{\omega}_a{}^{ij}(e) = 0 \\
\alpha &= v,\ (J,K)=(j,k): \quad \epsilon^{ab}\epsilon_{jk}\hat{D}_a\hat{e}^2_b = 0 = \hat{\omega}_{[a}{}^{2i}\hat{e}^i_{\ b]} \implies \hat{\omega}_a{}^{2i} = M^{ik}e_{ak} \equiv M^i_a \\
\alpha &= b,\ (J,K)=(j,k): \quad \epsilon^{ab}\epsilon_{jk}\hat{D}_{[a}\,\hat{e}^2_{\ v]} = 0 = \hat{\omega}_v{}^{2i}\hat{e}^i_a \implies \hat{\omega}_v{}^{2i} = 0 \\
\alpha &= b,\ (J,K)=(j,2): \quad \epsilon^{ab}\epsilon_{ij}\hat{D}_{[a}\,\hat{e}^i_{\ v]} = 0 = \hat{D}_v\hat{e}^i_a \implies \hat{\omega}_v{}^{ij} = -e^a_j\partial_v e^i_a
\end{aligned}$$

- Torsion is zero.
- Two metric(g_{ij}) and its determinant are independent of the degenerate direction.

$$\hat{\omega}_a{}^{ij} = \bar{\omega}_a{}^{ij}(e) \qquad \hat{\omega}_a{}^{2i} = M^{ik}e_{ak}$$

$$\hat{\omega}_v{}^{2i} = 0 \qquad \hat{\omega}_v{}^{ij} = 0$$

- The second set of equations :

$$\begin{aligned}
\alpha &= a, k = i: \quad \epsilon^{ab}\epsilon_{ij}\hat{R}_{va}{}^{2i} = 0 \implies \partial_v \hat{w}_a^{2i} = 0 = \partial_v M^{ij} \\
\alpha &= a, k = 2: \quad \epsilon^{ab}\epsilon_{ij}\hat{R}_{va}{}^{ij} = 0 \\
\alpha &= v, k = i: \quad \epsilon^{ab}\epsilon_{ij}\hat{R}_{ab}{}^{2i} = 0 = \bar{D}_{[a} M^i_{b]} \\
\alpha &= v, k = 2: \quad \epsilon^{ab}\epsilon_{ij}\left[\xi\hat{R}_{ab}{}^{ij} + \beta e^i_a e^j_b\right] = 0 = \epsilon^{ab}\epsilon_{ij}\left[\xi\bar{R}_{ab}{}^{ij}(\bar{\omega}) - 2\xi M^i_a M^j_b + \beta e^i_a e^j_b\right]
\end{aligned}$$

- M^{ij} is independent of ν
- Last two equations summarize the contents of this theory.

Summary:

$$\bar{R}(\bar{w}(e)) + \frac{\beta}{\xi} = M^2 - M^i_a M^a_i$$

$$\bar{D}_{[a} M^i_{b]} = 0$$

- Only metric components are dynamical.
- Second set of equations are constraints.
- For $M^i_a = \lambda e^i_a$

$$\bar{R} + \left[\frac{\beta}{\xi} - 2\lambda^2\right] = 0$$

- Special case reduces to Jackiw-Titelboim gravity upto an identification.

Degenerate EGB

- The degenerate EGB action is given by:

$$\mathcal{L}(\hat{e},\hat{\omega}) = \epsilon^{\mu\nu\alpha\beta\gamma}\epsilon_{IJKLM}\left[\frac{\alpha}{2}\hat{R}^{IJ}_{\mu\nu}(\hat{\omega})\hat{R}^{KL}_{\alpha\beta}(\hat{\omega})\hat{e}^{M}_{\gamma} + \frac{\zeta}{2}\hat{R}^{IJ}_{\mu\nu}(\hat{\omega})\hat{e}^{K}_{\alpha}\hat{e}^{L}_{\beta}\hat{e}^{M}_{\gamma} + \frac{\beta}{5}\hat{e}^{I}_{\mu}\hat{e}^{J}_{\nu}\hat{e}^{K}_{\alpha}\hat{e}^{L}_{\beta}\hat{e}^{M}_{\gamma}\right]$$

- The equations of motions are given by:

$$\epsilon^{\mu\nu\alpha\beta\gamma}\epsilon_{IJKLM}\left[\alpha\hat{R}^{IJ}_{\mu\nu}(\hat{\omega}) + \zeta\hat{e}^{I}_{\mu}\hat{e}^{J}_{\nu}\right]\hat{D}_{\alpha}(\hat{\omega})\hat{e}^{K}_{\beta} = 0$$

$$\epsilon^{\mu\nu\alpha\beta\gamma}\epsilon_{IJKLM}\left[\frac{\alpha}{2}\hat{R}^{IJ}_{\mu\nu}(\hat{\omega})\hat{R}^{KL}_{\alpha\beta}(\hat{\omega}) + \zeta\hat{R}^{IJ}_{\mu\nu}(\hat{\omega})\hat{e}^{K}_{\alpha}\hat{e}^{L}_{\beta} + \beta\hat{e}^{I}_{\mu}\hat{e}^{J}_{\nu}\hat{e}^{K}_{\alpha}\hat{e}^{L}_{\beta}\right] = 0$$

- Similar to the previous case we assume the following ansatz for the vielbeins fields:

$$\hat{e}^I_\upsilon = 0$$
$$\hat{e}^I_\mu = \begin{bmatrix} \hat{e}^i_a \equiv e^i_a & 0 \\ 0 & 0 \end{bmatrix} \tag{1}$$

- where υ is the degenerate direction
- The final equation of motion is given by:

Equations of Motion

$$\epsilon^{abcd}\epsilon_{ijkl}\left[\phi\bar{R}^{ij}_{ab}e^k_c e^l_d + \frac{\alpha}{2}\bar{R}^{ij}_{ab}\bar{R}^{kl}_{cd} + \chi e^i_a e^j_b e^k_c e^l_d\right] = 0$$

- Where we have chosen $M^i_a = \lambda\, e^i_a$
- $\phi = (\zeta - 2\alpha\lambda^2)$ and $\chi = (\beta - 2\zeta\lambda^2 + 2\alpha\lambda^4)$

Static spherically symmetric solution

- Let us take the following ansatz:

$$ds^2 = -e^{\mu(r)}dt^2 + \frac{dr^2}{e^{\mu(r)}} + r^2 d\Omega^2$$

- Plugging this into the equation of motion fixes the form of e^{μ}, which is

Solution

$$e^{\mu} = 1 + \frac{\phi}{2\alpha}r^2 \pm \frac{1}{2}\left[\left(\frac{\phi^2}{\alpha^2} - \frac{2\chi}{\alpha}\right)r^4 + 4C_1 r - \frac{4C_2}{\alpha}\right]^{\frac{1}{2}}$$

- where C_1 and C_2 are the constants of integration.

Physical Interpretation

- Asymtotic limit:

$$e^{\mu} \to 1 - \frac{\Lambda_{eff}}{3} \pm \left(\frac{2M_{eff}}{r} - \frac{Q^2_{eff}}{r^2} \right)$$

- $2M_{eff} = \frac{C_1}{\sqrt{\frac{\phi^2}{\alpha^2} - \frac{2\chi}{\alpha}}}$
- $Q^2_{eff} = \frac{C_2}{\alpha\sqrt{\frac{\phi^2}{\alpha^2} - \frac{2\chi}{\alpha}}}$
- $\Lambda_{eff} = \frac{-3}{2} \left(\frac{\phi}{\alpha} \pm \sqrt{\frac{\phi^2}{\alpha^2} - \frac{2\chi}{\alpha}} \right)$
- $C_1 < 0 (C_1 > 0)$ and $C_2 < 0 (C_2 > 0)$ for the $+(-)$ branches

Objective:

Can we obtain constraints on the parameters M , Q amd $\frac{\phi}{\alpha}$??

	Branch	Metric
Q = 0	$+$	$e^{\mu} = 1 - \frac{r^2}{2\kappa}\left(1 - \left(1 - \frac{8M\kappa}{r^3}\right)^{\frac{1}{2}}\right)$
	$-$	$e^{\mu} = 1 + \frac{r^2}{2\kappa}\left(1 - \left(1 + \frac{8M\kappa}{r^3}\right)^{\frac{1}{2}}\right)$
$Q \neq 0$	$+$	$e^{\mu} = 1 - \frac{r^2}{2\kappa}\left(1 - \left(1 - \frac{8M\kappa}{r^3} + \frac{4Q^2\kappa}{r^4}\right)^{\frac{1}{2}}\right)$
	$-$	$e^{\mu} = 1 + \frac{r^2}{2\kappa}\left(1 - \left(1 + \frac{8M\kappa}{r^3} - \frac{4Q^2\kappa}{r^4}\right)^{\frac{1}{2}}\right)$

Conclusion

The above metrics match the solutions obtained via standard EGB theories. However unlike the standard theories the positive branch is also well defined in this framework.

	Branch	Horizon	Allowed parameters
Q = 0	$+$	$r_{\pm} = M \pm \sqrt{M^2 + \kappa}$	$0 \leq \kappa \leq 8M^2$
	$-$	$r_{\pm} = M \pm \sqrt{M^2 - \kappa}$	$0 < \kappa \leq M^2$
Q $\neq$ 0	$+$	$r_{\pm} = M \pm \sqrt{M^2 - Q^2 + \kappa}$	$0 < \kappa \leq \frac{M^2}{2}, \quad Q^2 \leq 2\sqrt{2\kappa M^2} - \kappa$ $\frac{M^2}{2} < \kappa < 8M^2, \quad Q^2 \leq M^2 + \kappa$
	$-$	$r_{\pm} = M \pm \sqrt{M^2 - Q^2 - \kappa}$	$0 < \kappa \leq M^2,\ Q^2 \leq M^2 - \kappa$

Probing scalar QNMs

- propagation of massless scalar fields: $\Box\phi = 0$
- Ansatz: $\Phi(t, r, \theta, \phi) = Y(\theta, \phi)\frac{u(r)e^{-i\omega t}}{r}$
- The radial component of the Klein-Gordon equation

$$\frac{d^2u}{dr_*^2} + [\omega^2 - V_{eff}]u = 0$$

- r_* is the tortoise coordinate
- $V_{eff}(r) = e^{\mu}\left(\frac{\ell(\ell+1)}{r^2} + \frac{(e^{\mu})'}{r}\right)$
- $V_{eff}(r)$ has the imprints of the background spacetime.

- Asymptotic boundary conditions : $\frac{u(r)e^{-i\omega t}}{r} \sim e^{-i\omega(r_*+t)} \quad r_* \to -\infty$ and $\frac{u(r)e^{-i\omega t}}{r} \sim e^{i\omega(r_*-t)} \quad r_* \to \infty$
- Computation for discrete QNMs with single barrier potential: WKB method along with Pade improvements.
- To visualize the behaviour of the scalar field we also compute the time domain profiles.

Not the same as gravitational perturbation

Tetrad perturbations in the context of degenerate theories not well understood. Hence we probe stability of spacetime via propagation of massless scalar field.

Effective Potentials

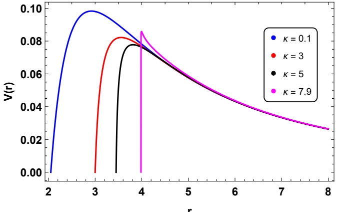


Figure: Positive Branch

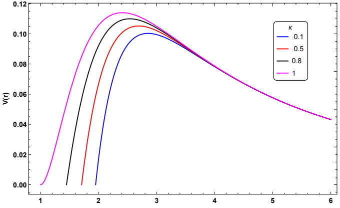


Figure: Negative Branch

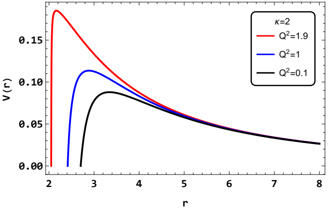


Figure: Positive Branch with charge

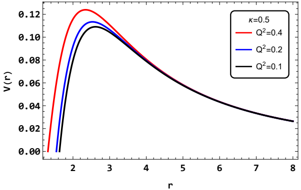


Figure: Negative branch with charge

Comparison with GR counterparts

ℓ	Schwarzschild	$\kappa = 0.001$	$\kappa = 0.01$	$\kappa = 0.1$
1	0.292909 -i 0.09776	0.292928 -i 0.097405	0.292692 -i 0.097818	0.290542 -i 0.099187
2	0.48364 -i 0.096757	0.483610 -i 0.096733	0.483274 -i 0.096908	0.480022 -i 0.0982177
3	0.675365 -i 0.096499	0.675436 -i 0.098474	0.674859 -i 0.096646	0.670409 -i 0.097922

Table: Comparison of fundamental ω_{QNM} for different κ values corresponding to the positive branch of $Q = 0$ case with that of the Schwarzschild black hole.

ℓ	Schwarzschild	$\kappa = 0.001$	$\kappa = 0.1$	$\kappa = 0.7$	$\kappa = 0.9$
1	0.292909 -i 0.09776	0.2931761 -i 0.0975015	0.295398 -i 0.096027	0.3122758 -i 0.083077	0.318868 -i 0.076427
2	0.48364 -i 0.096757	0.484015 -i 0.096608	0.487438 -i 0.0952057	0.51529 -i 0.082522	0.527432 -i 0.075695
3	0.675365 -i 0.096499	0.6758755 -i 0.096352	0.680578 -i 0.094978	0.719381 -i 0.082369	0.736788 -i 0.075478

Table: Comparison of fundamental ω_{QNM} for different κ values belonging to the negative branch with zero geometric charge with that of the Schwarzschild black hole.

Comparison with GR counterparts

ℓ	Q	Reissner-Nordström	$\kappa = 0.01$	$\kappa = 0.1$
1	0.1	0.293407 -0.0978114 i	0.293168 -0.0979535 i	0.291007 -0.0992218 i
	0.9	0.352625 -0.0972076 i	0.352036 -0.0972604 i	0.347043 -0.101713 i
2	0.1	0.484455 -0.0968185 i	0.484082 -0.0969688 i	0.480797 -0.0982871 i
	0.9	0.581952 -0.0966326 i	0.580974 -0.0971101 i	0.572645 -0.100969 i
3	0.1	0.676499 -0.0965534 i	0.675988 -0.096701 i	0.671501 -0.0979877 i
	0.9	0.812568 -0.0964701 i	0.811185 -0.0969426 i	0.799493 -0.100743 i

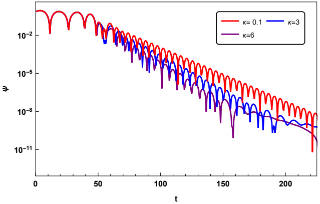
Table: Comparison of fundamental ω_{QNM} for different values of κ and Q with that of the Reissner-Nordström black hole.

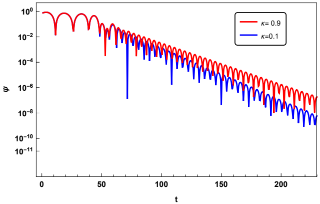
ℓ	Q	Reissner-Nordström	$\kappa = 0.01$	$\kappa = 0.1$
1	0.1	0.293407 -0.0978114 i	0.293645 -0.0976688 i	0.295797 -0.0963567 i
	0.9	0.352625 -0.0972076 i	0.35319 -0.0967101 i	0.358452 -0.0914129 i
2	0.1	0.484455 -0.0968185 i	0.484829 -0.0966674 i	0.488285 -0.0952389 i
	0.9	0.581952 -0.0966326 i	0.582942 -0.0961435 i	0.592422 -0.0910936 i
3	0.1	0.676499 -0.0965534 i	0.677012 -0.0964049 i	0.681755 -0.0950176 i
	0.9	0.812568 -0.0964701 i	0.81397 -0.0959856 i	0.827516 -0.0909645 i

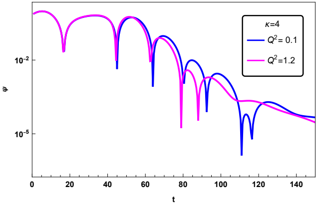
Table: Comparison of fundamental ω_{QNM} for different values of κ and Q with that of the Reissner-Nordström black hole.

Branch	κ	Q^2	Re (ω)	Im (ω)
$+$	Increasing	0	$\downarrow$	$\uparrow$
$-$	Increasing	0	$\uparrow$	$\downarrow$
$+$	Small	Increasing	$\uparrow$	$\uparrow$ to $\downarrow$
$+$	Large	Increasing	$\uparrow$	$\uparrow$
$-$	Fixed	Increasing	$\uparrow$	$\downarrow$

Table: QNMs' dependence on the metric parameters for different spacetimes. $\uparrow$ and $\downarrow$ indicate behaviour of the corresponding quantity with the change in the parameter.







Summary

- We have obtained the constraints on the parameter κ for the various asymptotically flat cases
- Performed stability analysis via propagation of massless scalar field.
- All the solutions are stable for the allowed parameter ranges.
- A definitive answer would require analysis of gravitational perturbations.
- Modes of our black holes converge towards their GR counterparts as κ tends to zero.
- More constraints on κ from shadow calculations
- Inclusion of cosmological constant.

Thank You