Primary Gravitational Waves at High Frequencies

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March 5th 2026, IIT Madras

Based on A. Hoory, JM, A. Paul and L. Sriramkumar: arXiv:2512.03959

Outline

- Primordial gravitational waves (PGWs): basics
- Calculation of the Power spectrum (PS) of PGWs
- Regulating the PGWs PS
- Smoothing the transitions
- The Born approximation
- Conclusions

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- PGWs correspond to the transverse and traceless part of the perturbed metric

$$\mathrm{d}s^2 = a^2(\eta)\left\{-\mathrm{d}\eta^2 + \left[\delta_{ij} + h_{ij}\left(\eta, \boldsymbol{x}\right)\right]\mathrm{d}x^i\mathrm{d}x^j\right\}$$

$$\delta^{ij}h_{ij}(\eta, \boldsymbol{x}) = 0$$

$$\delta^{ij}\partial_i h_{j\ell}(\eta, \boldsymbol{x}) = 0$$

- Equation of propagation

$$h_{ij}'' + 2\mathcal{H}h_{ij}' - \delta^{mn}\partial_m\partial_n h_{ij} = 0$$

This depends on the scale factor through the conformal Hubble parameter

$$\mathcal{H} = \frac{a'}{a}$$

- The PGWs source are the unavoidable vacuum quantum fluctuations

$$\hat{h}_{ij}(\eta, \boldsymbol{x}) = \frac{\sqrt{2}}{M_{\mathrm{P}\ell}a(\eta)} \sum_{\lambda=+,\times} \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}} \varepsilon_{ij}^{\lambda}(\boldsymbol{n}) \left[\hat{c}_{\boldsymbol{k}}^{\lambda}\mu_k(\eta)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \hat{c}_{\boldsymbol{k}}^{\lambda\dagger}\mu_k^*(\eta)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right]$$

- Annihilation and creation operators: $\left[\hat{c}_{\boldsymbol{k}}^{\lambda}, \hat{c}_{\boldsymbol{k}'}^{\lambda'\dagger}\right] = \delta^{(3)}\left(\boldsymbol{k} - \boldsymbol{k}'\right)\delta^{\lambda\lambda'}$

- Polarization tensor: $\delta^{im}\delta^{jn}\varepsilon_{mn}^{\lambda}(\boldsymbol{n})\varepsilon_{ij}^{\lambda'}(\boldsymbol{n}) = 2\delta^{\lambda\lambda'}$

- Rescaled Fourier mode function: $\mu_k(\eta)$

- Obeys the equation of a parametric oscillator:

$$\mu_k'' + \left(k^2 - \frac{a''}{a}\right)\mu_k = 0$$

- The observables are

  - The energy density of PGWs which can affects the expansion rate of the Universe

  - The two-point correlation functions (and higher functions) which can expressed as the power spectrum in Fourier space

$$\delta^{im}\delta^{jn}\left\langle 0\left|\hat{h}_{mn}(\eta,\boldsymbol{x})\hat{h}_{ij}(\eta,\boldsymbol{x})\right|0\right\rangle = \int_0^{+\infty}\frac{\mathrm{d}k}{k}\mathcal{P}_{\mathrm{T}}(k,\eta)$$

$$\mathcal{P}_{\mathrm{T}}(k,\eta) = \frac{8}{M_{\mathrm{P}\ell}^2}\frac{k^3}{2\pi^2}\frac{|\mu_k(\eta)|^2}{a^2(\eta)}$$

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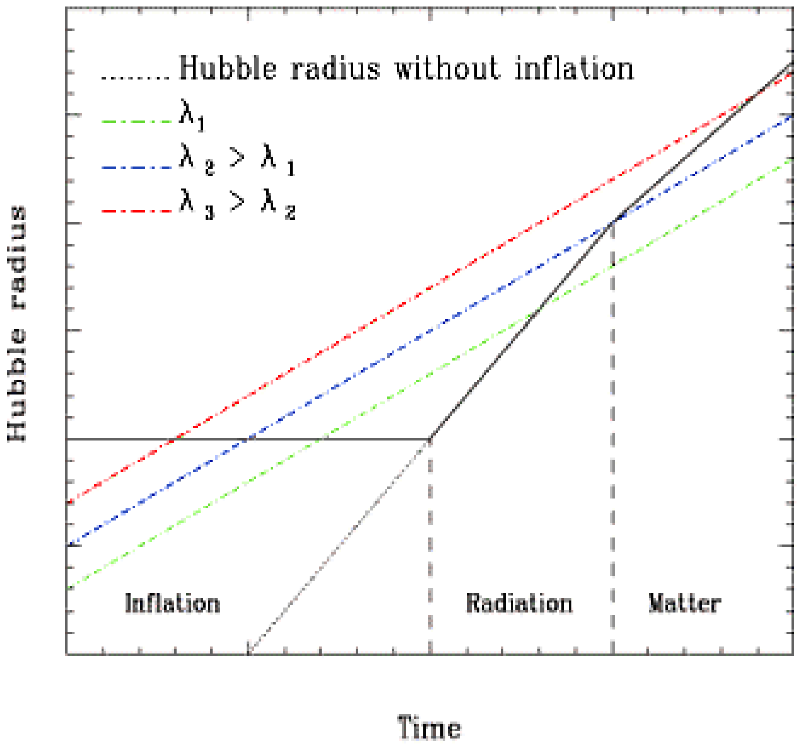
- Once the history of the Universe is known (ie the scale factor), the power spectrum can be calculated



$$a(\eta) = \begin{cases} -\dfrac{\ell_0}{\eta}, & \eta < \eta_{\rm end} < 0 \\ a_{\rm r}(\eta - \eta_{\rm r}), & \eta_{\rm end} < \eta < \eta_{\rm eq} \\ a_{\rm m}(\eta - \eta_{\rm m})^2, & \eta_{\rm eq} < \eta \end{cases}$$

- Matching condition at each transition

$$[a]_\pm = [a']_\pm = 0 \longrightarrow \begin{cases} a_{\rm r} = \dfrac{\ell_0}{\eta_{\rm end}^2} \\ \eta_{\rm r} = 2\eta_{\rm end} \end{cases} \longrightarrow \begin{cases} a_{\rm m} = \dfrac{a_{\rm r}}{2(\eta_{\rm eq} - \eta_{\rm m})} \\ \eta_{\rm m} = -\eta_{\rm eq} + 2\eta_{\rm r} = -\eta_{\rm eq} + 4\eta_{\rm end} \end{cases}$$

- During dS inflation, the Fourier mode function, normalized to the Bunch-Davies vaccum, is given by

$$\mu_k(\eta) = A_k y^{1/2} H_{3/2}^{(1)}(y) = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right) e^{-ik\eta}$$

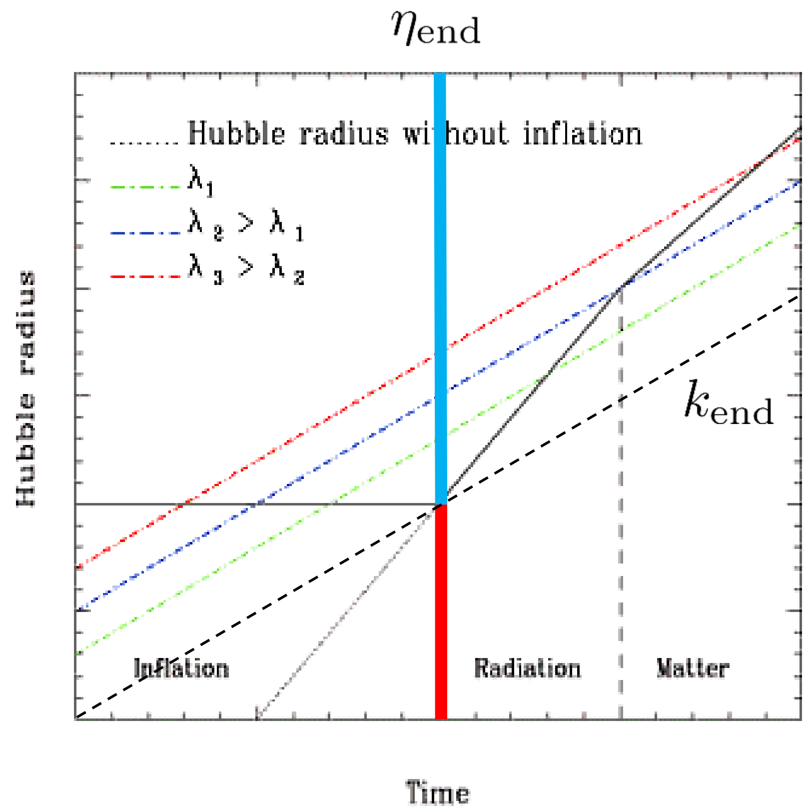
with $y = -k\eta > 0$

$$A_k = \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{2k}} e^{i\pi\mu/2 + i\pi/4 + ik\eta_{\rm ini}}$$

↓

$$\mathcal{P}_{\rm T}(k, \eta_{\rm end}) = \frac{2H_{\rm end}^2}{\pi^2 M_{{\rm P}\ell}^2}\left(1 + y_{\rm end}^2\right)$$

with $y_{\rm end} = \frac{k}{k_{\rm end}}$



- During the radiation-dominated era, the Fourier mode function can be expressed as

$$\mu_k(\eta) = \alpha_k^{\mathrm{r}}\, m_k(\eta) + \beta_k^{\mathrm{r}}\, n_k(\eta)$$

with
$$m_k(\eta) = x^{1/2} H_{1/2}^{(2)}(x)$$
$$n_k(\eta) = x^{1/2} H_{1/2}^{(1)}(x) = m_k^*(\eta)$$
$$x = k(\eta - \eta_{\mathrm{r}}) = \frac{k}{\mathcal{H}}$$

- The Bogoliubov coefficients satisfy: $|\alpha_k^{\mathrm{r}}|^2 - |\beta_k^{\mathrm{r}}|^2 = |A_k|^2$

↳
$$\mathcal{P}_{\mathrm{T}}(k,\eta) = \frac{2}{\pi^2}\left(\frac{H_{\mathrm{end}}}{M_{\mathrm{P}\ell}}\right)^2 \left[\frac{a_{\mathrm{end}}}{a(\eta)}\right]^2 y_{\mathrm{end}}^2 \left[1 + 2\left|\frac{\beta_k^{\mathrm{r}}}{A_k}\right|^2 - 2\,\mathrm{Re}\left(\frac{\alpha_k^{\mathrm{r}}\beta_k^{\mathrm{r}\,*}}{|A_k|^2}\right)\cos(2x) - 2\,\mathrm{Im}\left(\frac{\alpha_k^{\mathrm{r}}\beta_k^{\mathrm{r}\,*}}{|A_k|^2}\right)\sin(2x)\right]$$

Valid for any Bogoliubov coefficients

Power spectrum of PGWs

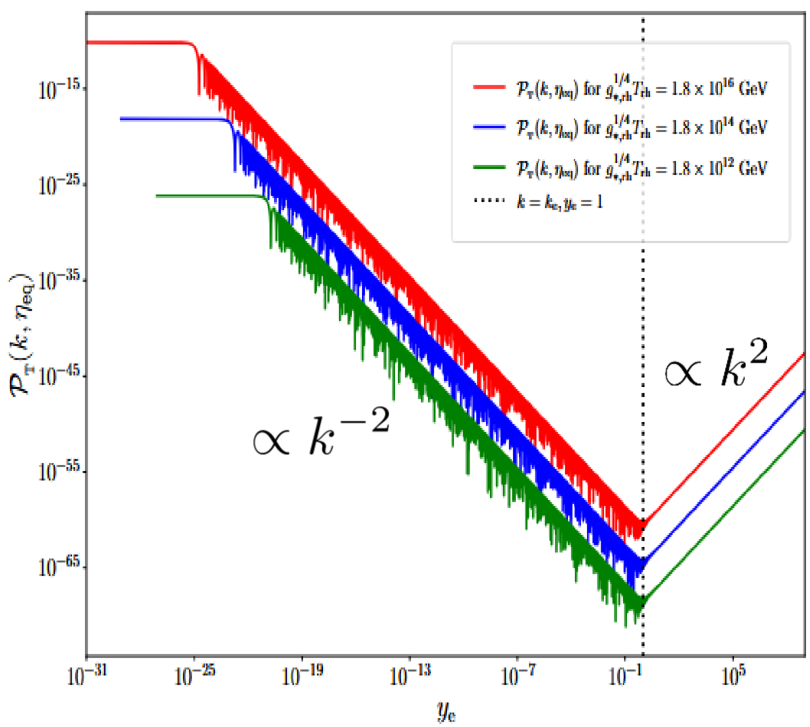
- The Bogoliubov coefficients can be calculated from the junction conditions

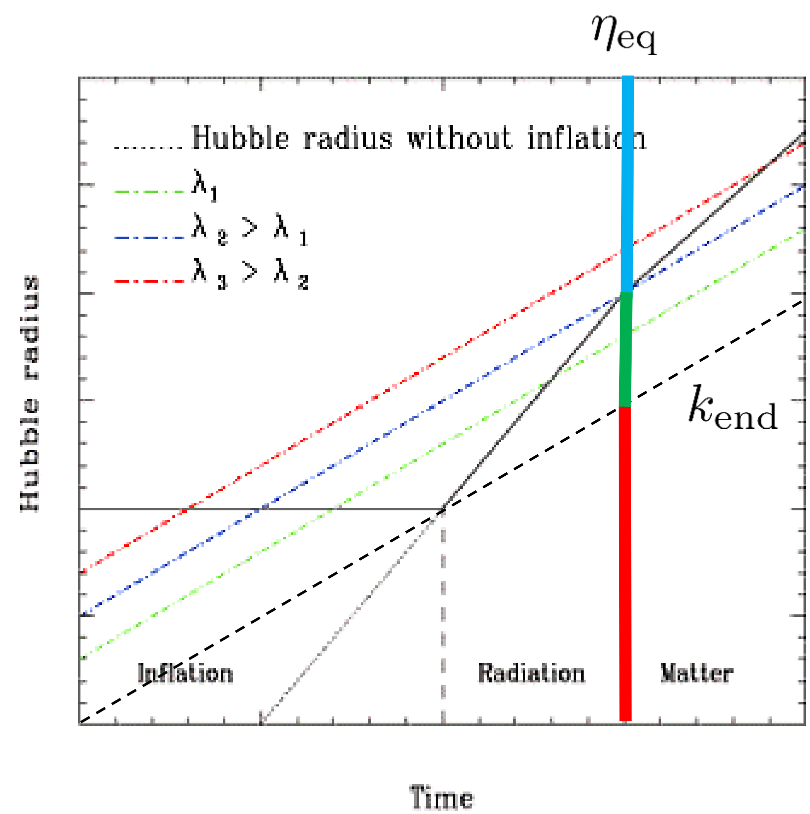
$$[\mu_k]_\pm = [\mu'_k]_\pm = 0$$

$$\downarrow$$

$$\alpha_k^{\rm r} = -\frac{iA_k}{2y_{\rm end}^2}\left(1 - 2iy_{\rm end} - 2y_{\rm end}^2\right)e^{2iy_{\rm end}}$$

$$\beta_k^{\rm r} = -\frac{iA_k}{2y_{\rm end}^2}$$





- During the matter-dominated era, the Fourier mode function can be expressed as

$$\mu_k(\eta) = \alpha_k^{\rm m} p_k(\eta) + \beta_k^{\rm m} q_k(\eta)$$

with $$p_k(\eta) = z^{1/2} H_{3/2}^{(2)}(z) = q_k^*(\eta)$$

$$z = k(\eta - \eta_{\rm m}) = 2\frac{k}{\mathcal{H}}$$

- The Bogoliubov coefficients satisfy: $|\alpha_k^{\rm m}|^2 - |\beta_k^{\rm m}|^2 = |A_k|^2$

$$\mathcal{P}_{\rm T}(k,\eta) = \frac{2}{\pi^2}\left(\frac{H_{\rm end}}{M_{\rm P\ell}^2}\right)^2 \left(\frac{a_{\rm end}}{a}\right)^2 y_{\rm end}^2 \left\{\left(1 + 2\left|\frac{\beta_k^{\rm m}}{A_k}\right|^2\right)\left(1+\frac{1}{z^2}\right) + 2\left(1-\frac{1}{z^2}\right)\left[{\rm Re}\left(\frac{\alpha_k^{\rm m}\beta_k^{\rm m *}}{|A_k|^2}\right)\cos(2z) + {\rm Im}\left(\frac{\alpha_k^{\rm m}\beta_k^{\rm m *}}{|A_k|^2}\right)\sin(2z)\right] - \frac{4}{z}\left[{\rm Re}\left(\frac{\alpha_k^{\rm m}\beta_k^{\rm m *}}{|A_k|^2}\right)\sin(2z) - {\rm Im}\left(\frac{\alpha_k^{\rm m}\beta_k^{\rm m *}}{|A_k|^2}\right)\cos(2z)\right]\right\}$$

Valid for any Bogoliubov coefficients

Power spectrum of PGWs

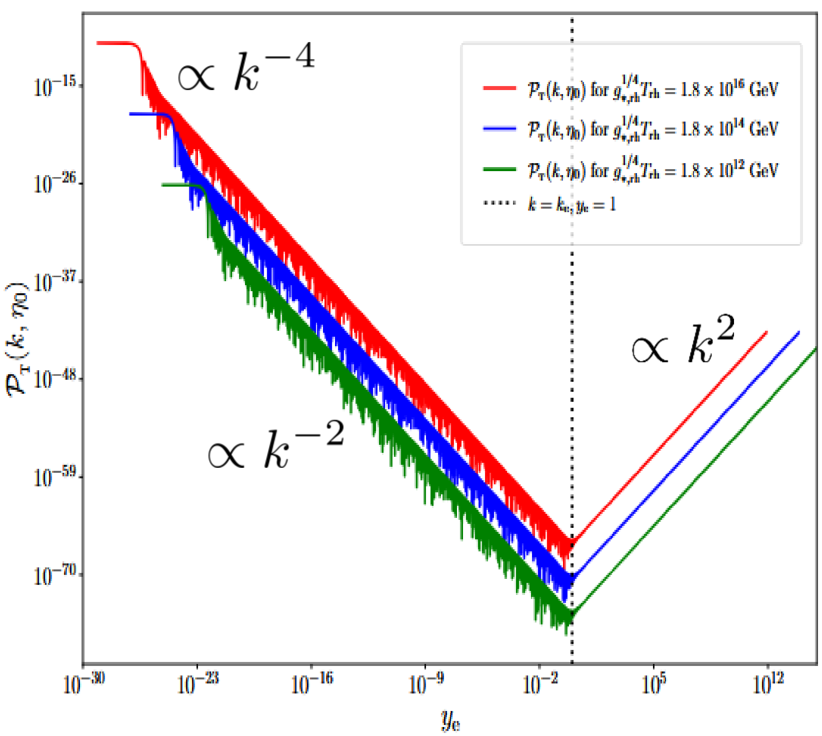
- The Bogoliubov coefficients can be calculated from the junction conditions

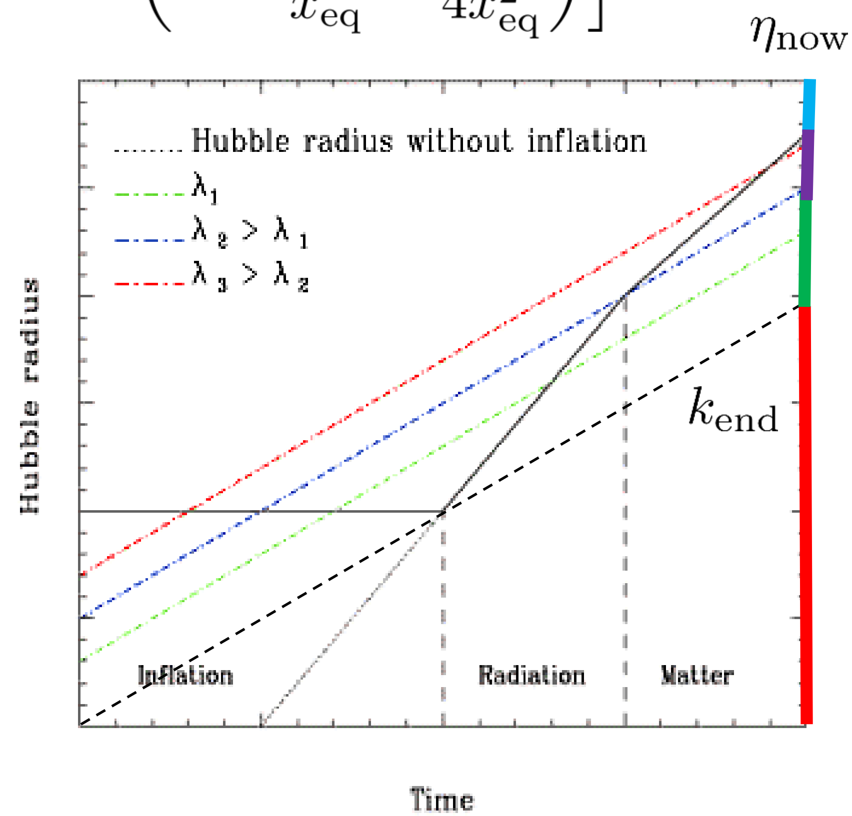
$$[\mu_k]_\pm = [\mu_k']_\pm = 0$$

$$\downarrow$$

$$\alpha_k^{\rm m} = \frac{e^{ix_{\rm eq}}}{2}\left[\alpha_k^{\rm r}\left(-2i + \frac{1}{x_{\rm eq}} + \frac{i}{4x_{\rm eq}^2}\right) - \frac{i\beta_k^{\rm r}}{4x_{\rm eq}^2}e^{2ix_{\rm eq}}\right],$$

$$\beta_k^{\rm m} = \frac{e^{-ix_{\rm eq}}}{2}\left[\frac{i\alpha_k^{\rm r}}{4x_{\rm eq}^2}e^{-2ix_{\rm eq}} + \beta_k^{\rm r}\left(2i + \frac{1}{x_{\rm eq}} - \frac{i}{4x_{\rm eq}^2}\right)\right]$$





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Problems in the high frequency regime:

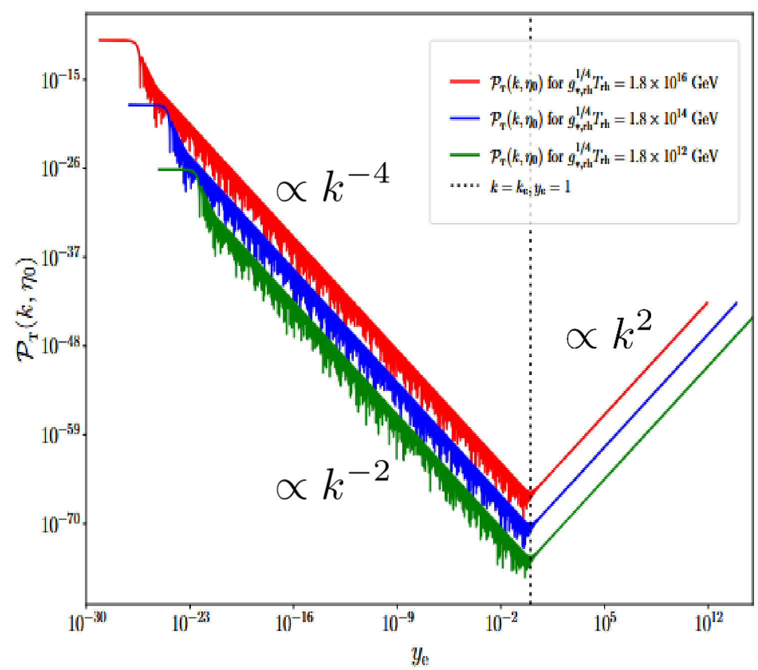
- Leads to divergent physical quantities

$$\hat{\mathcal{S}}(t) = \underbrace{D^{ij}}_{\text{Detector tensor}}\hat{h}_{ij}(t)$$

Detector tensor

$$\left\langle \hat{\mathcal{S}}^2(\eta) \right\rangle = \frac{F}{16\pi}\int_0^{+\infty}\frac{\mathrm{d}k}{k}\mathcal{P}_{\mathrm{T}}(k,\eta)$$

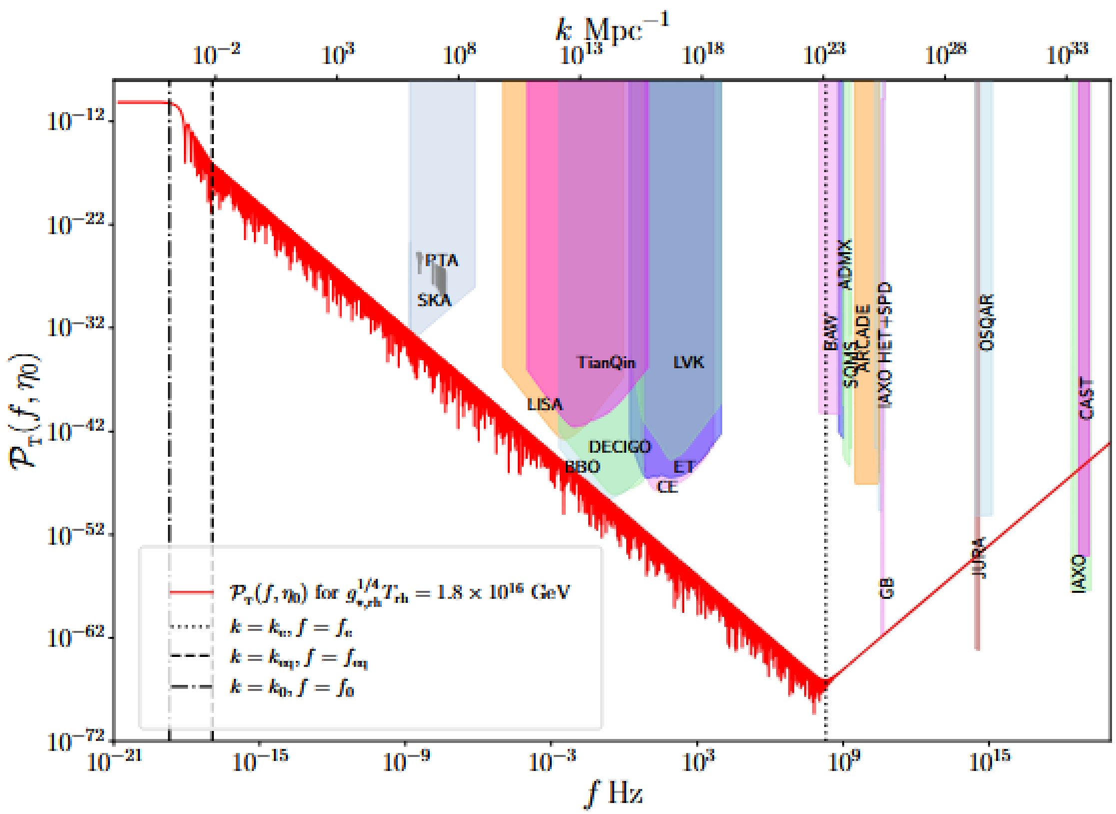
$$\left\langle \hat{\mathcal{S}}^2(\eta) \right\rangle = \infty$$



Problems in the high frequency regime:

- Leads to divergent physical quantities

- The growth proportional to k^2 should have already been detected



Problems in the high frequency regime:

- Leads to divergent physical quantities
- The growth proportional to k^2 should have already been detected

↓

This indicates the need to regularize the power spectrum

$$\mu_k''(\eta) + \left(k^2 - \frac{a''}{a}\right)\mu_k(\eta) = 0$$

↓

$$\mu_k^{\rm ad}(\eta) = \frac{1}{\sqrt{2W_k(\eta)}} \exp\left[-i\int_{\eta_{\rm ini}}^{\eta} {\rm d}\tilde{\eta}\, W_k(\tilde{\eta})\right]$$

↓

$$W_k^2(\eta) = k^2 - \frac{a''}{a} - \frac{1}{2}\left[\frac{W_k''}{W_k} - \frac{3}{2}\left(\frac{W_k'}{W_k}\right)^2\right]$$

Can be solved with an adiabatic expansion, ie an expansion in number of time derivatives

Number of time derivatives (→ n)

$$W_k(\eta) = \sum_{n=0}^{+\infty} W_k^{(n)}(\eta)$$

$$W_k^2 = k^2 \longrightarrow W_k^{(0)} = k$$

$$W_k^2 = k^2 - \frac{a''}{a} \longrightarrow W_k^{(2)} = -\frac{a''}{2ak}$$

Adiabatic subtraction

- The second order is sufficient to remove divergences in the two-point correlation function

$$|\mu_k^{\mathrm{ad}}|^2 = \frac{1}{2W_k}$$

$$\downarrow$$

$$\mathcal{P}_{_\mathrm{T}}^{\mathrm{ad}}(k,\eta) = \frac{2}{\pi^2}\left(\frac{H_{\mathrm{end}}}{M_{_{\mathrm{P}\ell}}}\right)^2 \left[\frac{a_{\mathrm{end}}}{a(\eta)}\right]^2 \underbrace{y_{\mathrm{end}}^2 \left(1\right.}_{\text{behaves as } k^2} \left. + \frac{1}{2k^2}\frac{a''}{a}\right)$$

$$\downarrow$$

$$\mathcal{P}_{_\mathrm{T}}^{\mathrm{reg}}(k,\eta) = \mathcal{P}_{_\mathrm{T}}(k,\eta) - \mathcal{P}_{_\mathrm{T}}^{\mathrm{ad}}(k,\eta)$$

Adiabatic subtraction in real space

- The real space correlation function at different points is

$$\delta^{im}\delta^{jn}\left\langle 0\left|\hat{h}_{mn}(\eta,\boldsymbol{x})\hat{h}_{ij}(\eta,\boldsymbol{x}')\right|0\right\rangle = \int_0^\infty \frac{\mathrm{d}k}{k}\frac{\sin(k|\boldsymbol{x}-\boldsymbol{x}'|)}{k|\boldsymbol{x}-\boldsymbol{x}'|}\mathcal{P}_{_\mathrm{T}}(k,\eta)$$

During the radiation-dominated era

$$\delta^{im}\delta^{jn}\left\langle 0\left|\hat{h}_{mn}(\eta,\boldsymbol{x})\hat{h}_{ij}(\eta,\boldsymbol{x}')\right|0\right\rangle^{\mathrm{ad}} = \int_0^\infty \frac{\mathrm{d}k}{k}\frac{\sin(k|\boldsymbol{x}-\boldsymbol{x}'|)}{k|\boldsymbol{x}-\boldsymbol{x}'|}\mathcal{P}^{\mathrm{ad}}_{_\mathrm{T}}(k,\eta)$$

$$= \frac{2}{\pi^2 M^2_{_{\mathrm{P}\ell}}}\frac{1}{|\boldsymbol{x}_{_\mathrm{P}}-\boldsymbol{x}'_{_\mathrm{P}}|^2}$$

Adiabatic subtraction corresponds to removing the flat spacetime correlation function

- The second order is sufficient to remove divergences in the two-point correlation function

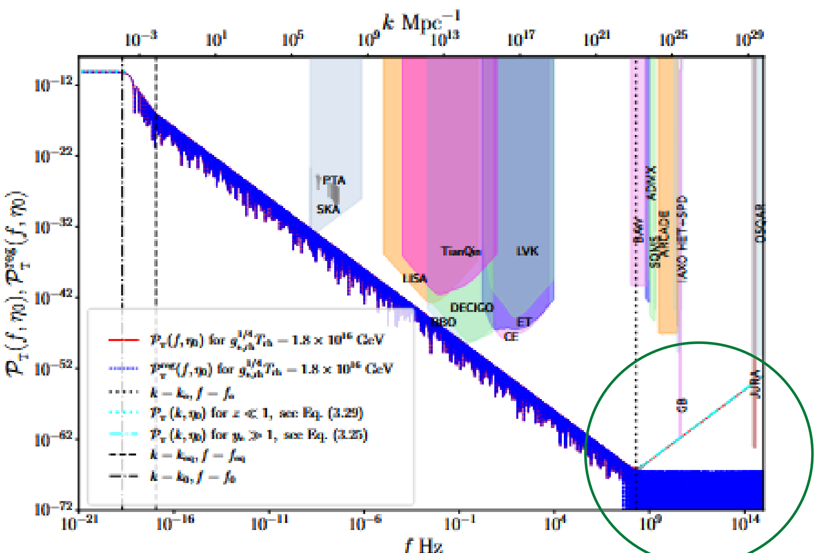
$$|\mu_k^{\rm ad}|^2 = \frac{1}{2W_k}$$

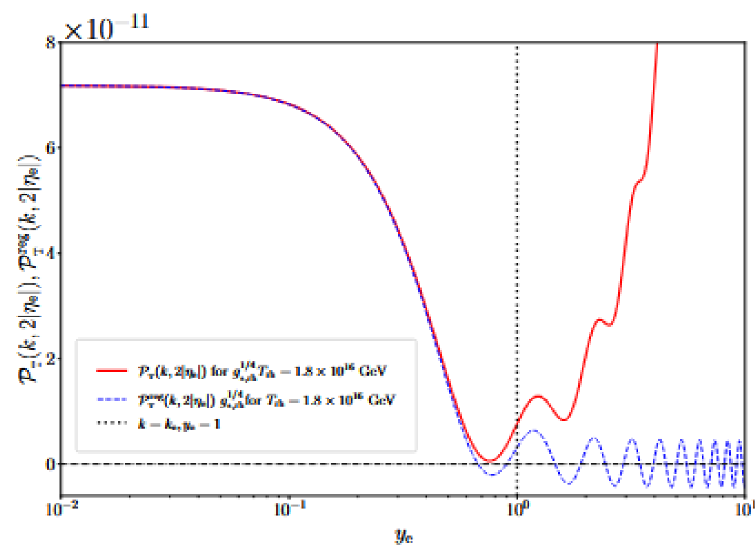
$$\downarrow$$

$$\mathcal{P}_{\rm T}^{\rm ad}(k,\eta) = \frac{2}{\pi^2}\left(\frac{H_{\rm end}}{M_{\rm P\ell}}\right)^2 \left[\frac{a_{\rm end}}{a(\eta)}\right]^2 \underbrace{y_{\rm end}^2 \left(1 + \frac{1}{2k^2}\frac{a''}{a}\right)}_{\text{behaves as } k^2}$$

$$\downarrow$$

$$\mathcal{P}_{\rm T}^{\rm reg}(k,\eta) = \mathcal{P}_{\rm T}(k,\eta) - \mathcal{P}_{\rm T}^{\rm ad}(k,\eta)$$





The k^2 rise is removed

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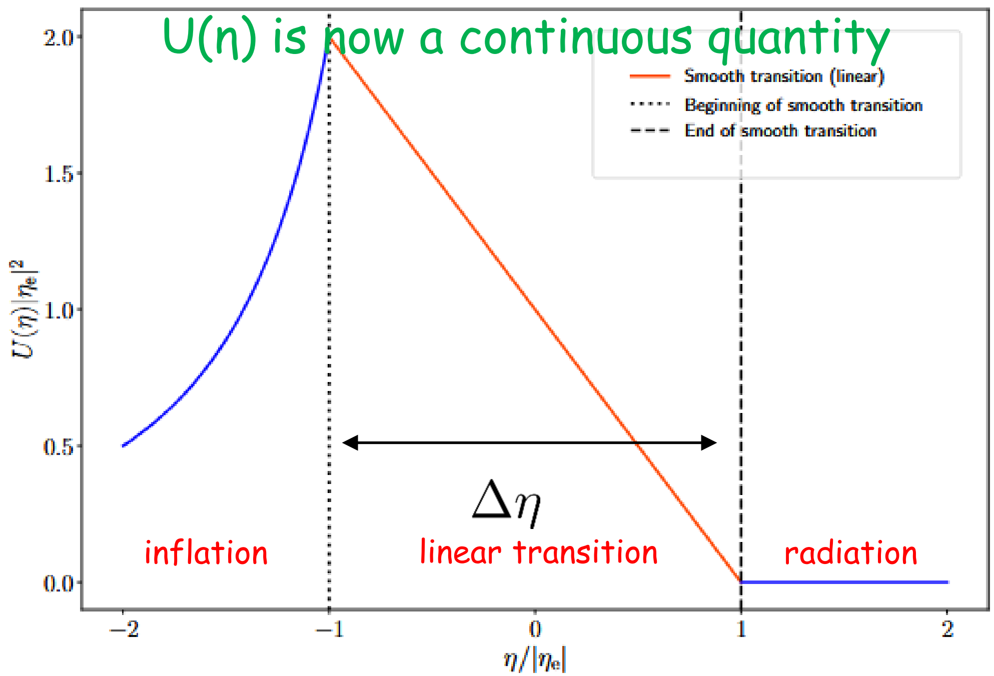
- The smoothness of the transitions can also affect the high-frequency regime of the PS -> let us smooth out the inflation-to-radiation transition



$$U(\eta) = \frac{a''}{a} = \begin{cases} \frac{2}{\eta^2}, & \eta < \eta_{\rm end} \\ \frac{2}{\eta_{\rm end}^2 \Delta\eta}\left[-(\eta - \eta_{\rm end}) + \Delta\eta\right], & \eta_{\rm end} < \eta < \eta_{\rm end} + \Delta\eta \\ 0, & \eta > \eta_{\rm end} + \Delta\eta \end{cases}$$

Smooth transition: scale factor

- Inflation:

$$a(\eta) = -\frac{\ell_0}{\eta}$$

Smooth transition: scale factor

- Inflation: $$a(\eta) = -\frac{\ell_0}{\eta}$$

- Linear transition: $$\frac{a(\eta)}{a_{\rm end}} = A_1 \mathrm{Ai}\left[\left|\frac{\mathrm{d}U}{\mathrm{d}\eta}\right|^{-2/3} U\right] + A_2 \mathrm{Bi}\left[\left|\frac{\mathrm{d}U}{\mathrm{d}\eta}\right|^{-2/3} U\right]$$

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- Radiation: $$a(\eta) = a_{\rm r}(\eta - \eta_{\rm r})$$

Smooth transition: scale factor

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- Radiation: $$a(\eta) = a_{\rm r}(\eta - \eta_{\rm r})$$

$$[a]_\pm = [a']_\pm = 0$$

$$a_{\rm r} = \frac{2^{1/3}}{3^{1/3}\Gamma(1/3)|\eta_{\rm end}|}\left(\frac{\Delta\eta}{|\eta_{\rm end}|}\right)^{-1/3}\left(A_1 - \sqrt{3}A_2\right)a_{\rm end} \xrightarrow{\Delta\eta \to 0} a_{\rm r} = \frac{\ell_0}{\eta_{\rm end}^2}$$

$$\frac{\eta_{\rm r}}{|\eta_{\rm end}|} = \frac{\Delta\eta}{|\eta_{\rm end}|} - 1 - \left(\frac{\Delta\eta}{|\eta_{\rm end}|}\right)^{1/3}\frac{\Gamma(1/3)}{6^{1/3}\Gamma(2/3)}\left(\frac{A_1 + \sqrt{3}A_2}{A_1 - \sqrt{3}A_2}\right) \longrightarrow \eta_{\rm r} = 2\eta_{\rm end}$$

Smooth transition: mode function

- Inflation: $$\mu_k(\eta) = A_k y^{1/2} H^{(1)}_{3/2}(y)$$

Smooth transition: mode function

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$$\mu_k(\eta) = A_k y^{1/2} H^{(1)}_{3/2}(y)$$

- Linear transition:

$$\mu_k(\eta) = B_1 \mathrm{Ai}[\tau(\eta)] + B_2 \mathrm{Bi}[\tau(\eta)]$$

$$\tau(\eta) = -\left|\frac{\mathrm{d}U}{\mathrm{d}\eta}\right|^{-2/3} \left[k^2 - U(\eta)\right]$$

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$$\mu_k(\eta) = \alpha^{\mathrm{r}}_k\, m_k(\eta) + \beta^{\mathrm{r}}_k\, n_k(\eta)$$

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- Radiation: $$\mu_k(\eta) = \alpha_k^{\mathrm{r}}\, m_k(\eta) + \beta_k^{\mathrm{r}}\, n_k(\eta)$$

$$[\mu_k]_\pm = [\mu_k']_\pm = 0$$

$$\alpha_k^{\mathrm{r}} \simeq i A_k e^{i(1-Y)y_{\mathrm{end}}} e^{i(\zeta_{\mathrm{end}}-\zeta_{\mathrm{tr}})}$$

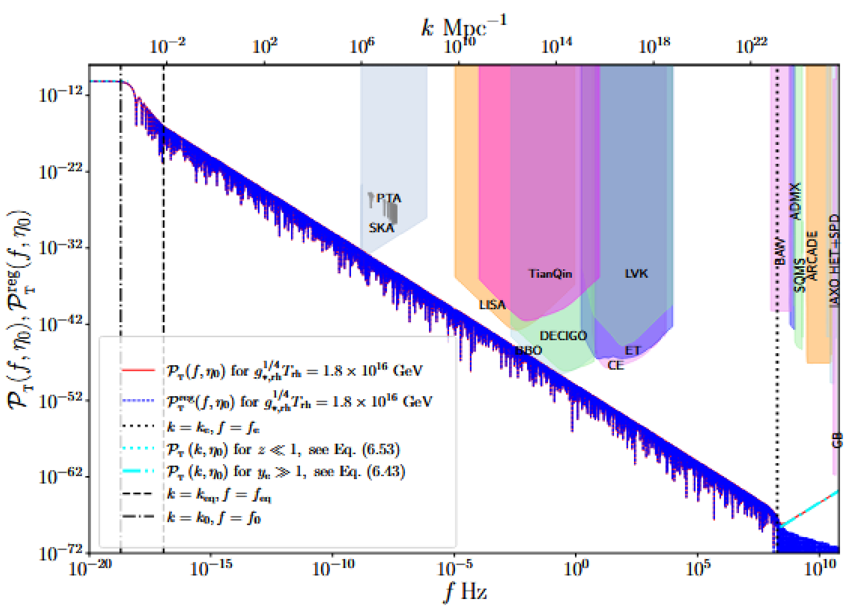
$$\beta_k^{\mathrm{r}} \simeq -\frac{A_k}{2y_{\mathrm{end}}^3} e^{i(1+Y)y_{\mathrm{end}}} e^{-i(\zeta_{\mathrm{end}}-\zeta_{\mathrm{tr}})}\left[1 + e^{i(\zeta_{\mathrm{end}}-\zeta_{\mathrm{tr}})}\frac{iX}{2}\sin\left(\zeta_{\mathrm{end}}-\zeta_{\mathrm{tr}}\right)\right]$$

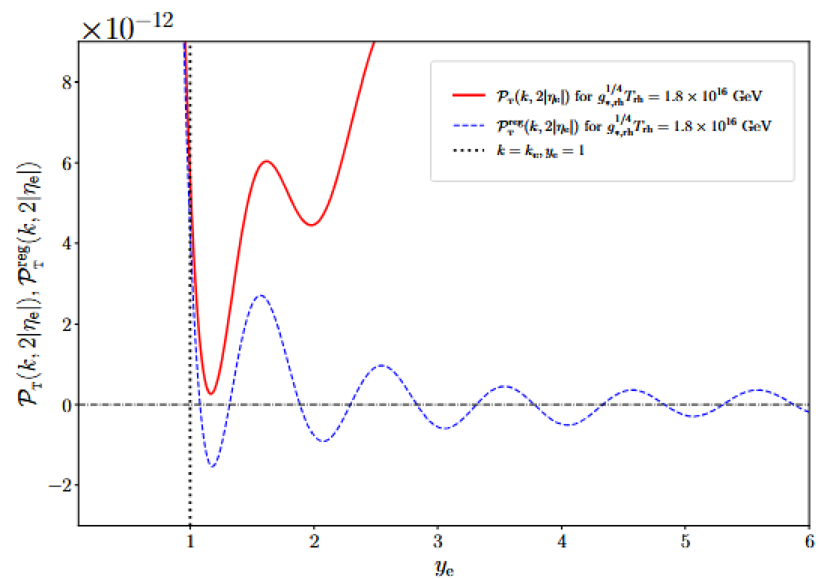
Smooth transition: mode function

$$\alpha_k^{\mathrm{r}} \simeq i A_k e^{i(1-Y)y_{\mathrm{end}}} e^{i(\zeta_{\mathrm{end}}-\zeta_{\mathrm{tr}})}$$

$$\beta_k^{\mathrm{r}} \simeq -\frac{A_k}{2y_{\mathrm{end}}^3} e^{i(1+Y)y_{\mathrm{end}}} e^{-i(\zeta_{\mathrm{end}}-\zeta_{\mathrm{tr}})} \left[1 + e^{i(\zeta_{\mathrm{end}}-\zeta_{\mathrm{tr}})} \frac{iX}{2} \sin\left(\zeta_{\mathrm{end}}-\zeta_{\mathrm{tr}}\right)\right]$$

scales as k^{-3} instead of k^{-2} (recall that $y_{\mathrm{end}} = \frac{k}{k_{\mathrm{end}}}$)





scales as k^{-1} instead of k^{0}

$$\alpha_k^{\rm r} \simeq i A_k e^{i(1-Y)y_{\rm end}} e^{i(\zeta_{\rm end}-\zeta_{\rm tr})}$$

$$\beta_k^{\rm r} \simeq -\frac{A_k}{2y_{\rm end}^3} e^{i(1+Y)y_{\rm end}} e^{-i(\zeta_{\rm end}-\zeta_{\rm tr})} \left[1 + e^{i(\zeta_{\rm end}-\zeta_{\rm tr})} \frac{iX}{2} \sin\left(\zeta_{\rm end}-\zeta_{\rm tr}\right)\right]$$

Abrupt transition limit and high-frequency limit: $\Delta\eta \to 0, \quad |\eta_{\rm end}| \sim {\rm finite}, \quad y_{\rm end} \gg 1$

$$X \equiv \frac{2|\eta_{\rm end}|}{\Delta\eta} \to \infty$$

$$\zeta_{\rm end} - \zeta_{\rm tr} \equiv \frac{2}{3}\frac{y_{\rm end}^3}{X}\left[\left(1 - \frac{2}{y_{\rm end}^2}\right)^{3/2} - 1\right] \to -\frac{2y_{\rm end}}{X} \to 0$$

$$\frac{iX}{2}\sin\left(\zeta_{\rm end}-\zeta_{\rm tr}\right) \to -i y_{\rm end}$$

$$Y \equiv 1 - \frac{\Delta\eta}{|\eta_{\rm end}|} + \frac{\eta_{\rm r}}{|\eta_{\rm end}|} \to -1$$ since $\eta_{\rm r} \to 2\eta_{\rm end}$

$$\alpha_k^{\rm r} \simeq i A_k e^{2iy_{\rm end}}, \quad \beta_k^{\rm r} \simeq -\frac{iA_k}{2y_{\rm end}^2}$$

One recovers the abrupt transition limit!

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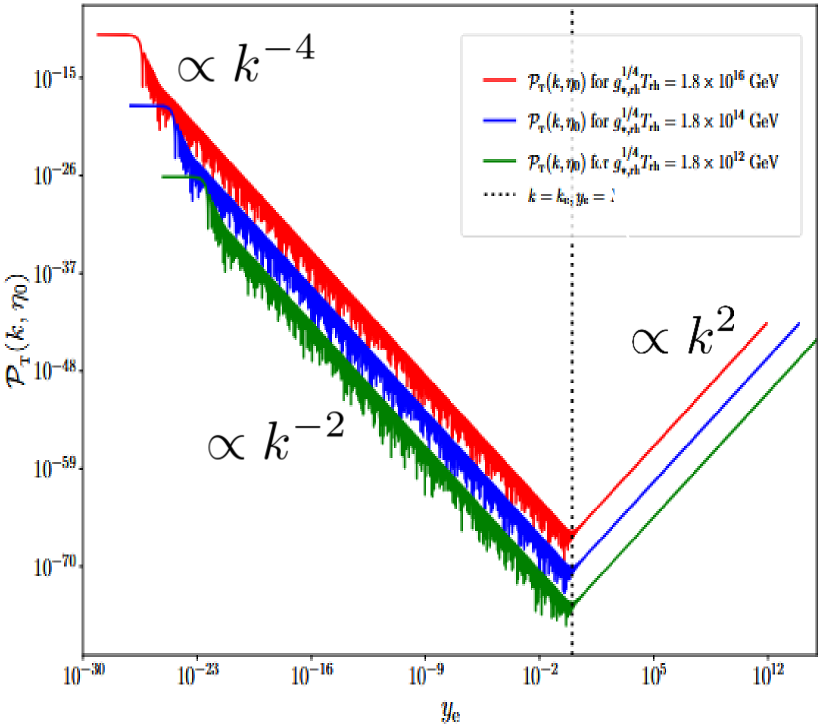
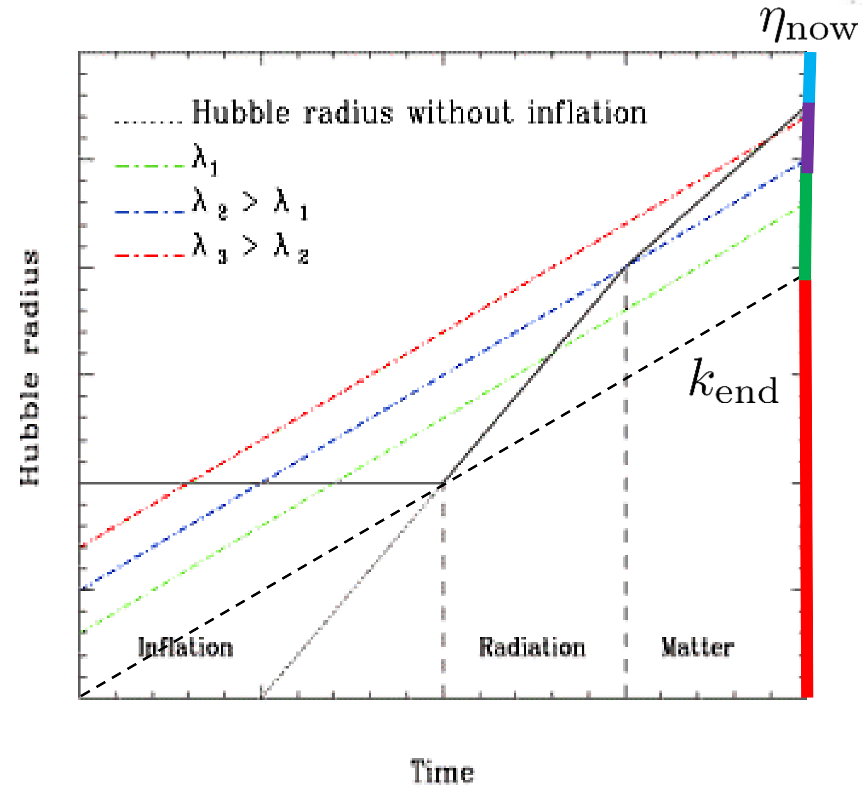
Born approximation





$$\mu_k'' + \mu_k\left[k^2 - U(\eta)\right] = 0$$

- One is interested in modes $k \gg k_{\mathrm{end}}$
- The max of the effective potential is $U_{\max} = U(\eta_{\mathrm{end}}) \sim \dfrac{1}{\eta_{\mathrm{end}}^2} \sim k_{\mathrm{end}}^2$

→ In this regime, one can treat U(η) as a perturbation

Born approximation

$$\mu_k'' + \mu_k \left[k^2 - U(\eta)\right] = 0$$

$$U(\eta) = 0 \longrightarrow \quad \mu_k^0 = \frac{1}{\sqrt{2k}} e^{-ik(\eta - \eta_{\rm ini})} = -A_k \sqrt{\frac{2}{\pi}} e^{-ik\eta}$$

$$U(\eta) \neq 0 \longrightarrow \quad \mu_k(\eta) \simeq \mu_k^0(\eta) + \int_{-\infty}^{+\infty} {\rm d}\bar{\eta}\, G(\eta, \bar{\eta})\, U(\bar{\eta})\, \mu_k^0(\bar{\eta})$$

with $$\frac{{\rm d}G(\eta,\bar{\eta})}{{\rm d}\eta^2} + k^2 G(\eta,\bar{\eta}) = \delta(\eta - \bar{\eta})$$

$$\longrightarrow \quad G(\eta,\bar{\eta}) = \Theta(\eta - \bar{\eta}) \frac{1}{k} \sin\left[k\left(\eta - \bar{\eta}\right)\right]$$

Born approximation

$$\mu_k'' + \mu_k \left[k^2 - U(\eta)\right] = 0$$

- The mode function is given by

$$\mu_k^{\mathrm{r}}(\eta) = \alpha_k m_k(\eta) + \beta_k n_k(\eta) = i\sqrt{\frac{2}{\pi}}\left[\alpha_k(\eta) e^{-ik(\eta-\eta_{\mathrm{r}})} - \beta_k(\eta) e^{ik(\eta-\eta_{\mathrm{r}})}\right]$$

↓

$$\alpha_k = iA_k\left[1 + \frac{i}{2k}\int_{-\infty}^{\eta} \mathrm{d}\bar{\eta}\, U(\bar{\eta})\right] e^{-ik\eta_{\mathrm{r}}}$$

$$\beta_k = -\frac{A_k}{2k}\left[\int_{-\infty}^{\eta} \mathrm{d}\bar{\eta}\, U(\bar{\eta})\, e^{-2ik\bar{\eta}}\right] e^{ik\eta_{\mathrm{r}}}$$

Born approximation and the instantaneous transition

$$U(\eta) = \frac{a''}{a} = \begin{cases} \dfrac{2}{\eta^2}, & \eta < \eta_{\rm end} \\ 0, & \eta > \eta_{\rm end} \end{cases}$$

↓

$$\beta_k = -\frac{A_k}{k}\left[\int_{-\infty}^{\eta_{\rm end}} \frac{\mathrm{d}\bar{\eta}}{\bar{\eta}^2} e^{-2ik\bar{\eta}}\right] e^{2ik\eta_{\rm end}}$$

↓

$$\beta_k = \frac{A_k}{y_{\rm end}} \sum_{n=1}^{+\infty} \frac{n!}{(2iy_{\rm end})^n} = -\frac{iA_k}{2y_{\rm end}^2} + \cdots$$

One recovers the exact calculation and the conclusion that $\beta_k \sim k^{-2}$

Born approximation and the linear transition

$$U(\eta)=\frac{a''}{a}=\begin{cases}\dfrac{2}{\eta^2}, & \eta<\eta_{\rm end}\\[2mm] \dfrac{2}{\eta_{\rm end}^2\Delta\eta}\left[-(\eta-\eta_{\rm end})+\Delta\eta\right], & \eta_{\rm end}<\eta<\eta_{\rm end}+\Delta\eta\\[2mm] 0, & \eta>\eta_{\rm end}+\Delta\eta\end{cases}$$

$$\downarrow$$

$$\beta_k=\frac{A_k}{y_{\rm end}}e^{iy_{\rm end}(2+\eta_{\rm r}/|\eta_{\rm end}|)}\left[\sum_{n=2}^{+\infty}\frac{n!}{(2iy_{\rm end})^n}+\frac{1}{4y_{\rm end}^2k_{\rm end}\Delta\eta}\left(e^{-2iy_{\rm end}k_{\rm end}\Delta\eta}-1\right)\right]$$

$$\to-\frac{A_k}{2y_{\rm end}^3}e^{iy_{\rm end}(2+\eta_{\rm r}/|\eta_{\rm end}|)}\left[1-\frac{1}{2k_{\rm end}\Delta\eta}\left(e^{-2iy_{\rm end}k_{\rm end}\Delta\eta}-1\right)\right]$$

One recovers the exact calculation and the conclusion that $\beta_k \sim k^{-3}$

Born approximation

- The advantage is that we can calculate the Bogoliubov for more complicated transitions (provided one can calculate the corresponding integral)

$$\beta_k = -\frac{A_k}{2k}\left[\int_{-\infty}^{\eta} \mathrm{d}\bar{\eta}\, U(\bar{\eta})\, e^{-2ik\bar{\eta}}\right] e^{ik\eta_{\mathrm{r}}}$$

- Quadratic $U(\eta) = c_0 + c_1\eta + \frac{c_2}{2}\eta^2$ $\longrightarrow$ $\beta_k \sim k^{-3}$

- Cubic $U(\eta) = c_0 + c_1\eta + \frac{c_2}{2}\eta^2 + c_3\frac{\eta^3}{3}$ $\longrightarrow$ $\beta_k \sim k^{-4}$

- Quintic $U(\eta) = c_0 + c_1\eta + \frac{c_2}{2}\eta^2 + c_3\frac{\eta^3}{3} + c_4\frac{\eta^4}{4} + c_5\frac{\eta^5}{5}$ $\longrightarrow$ $\beta_k \sim k^{-5}$

etc ...

Born approximation

- Infinitely smooth transition to interpolate between de Sitter and radiation

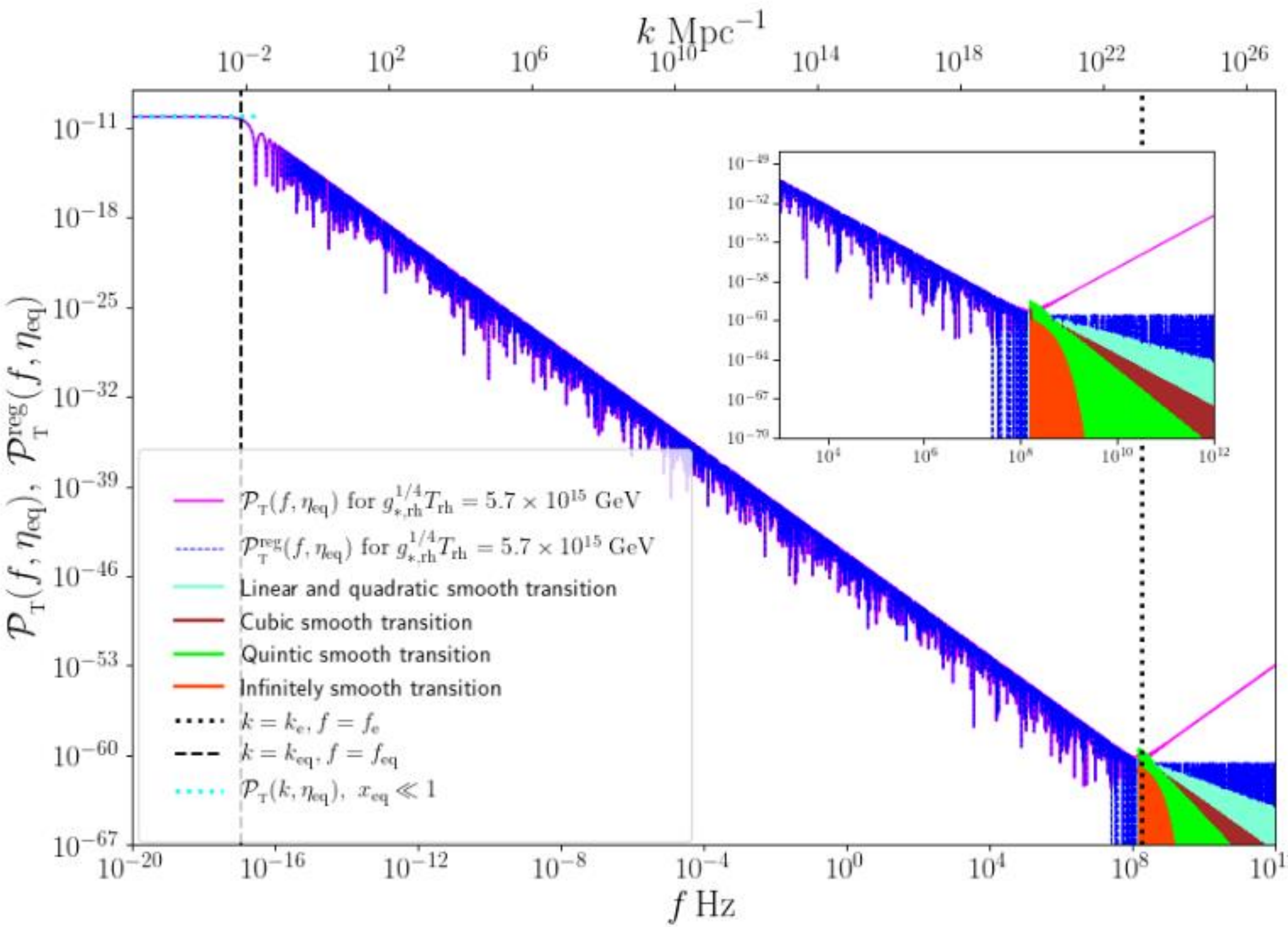
$$U(\eta) = \frac{1}{\eta^2 + (\gamma_{\rm end}\eta_{\rm end})^2}\left[1 - \tanh\left(\frac{\eta - \eta_{\rm end}}{\Delta\eta_{\rm end}}\right)\right]$$

↓

$$\beta_k = -\frac{\pi A_k}{2\gamma_{\rm end} y_{\rm end}} \underbrace{e^{-2\gamma_{\rm end} y_{\rm end}}} \left[1 - \tanh\left(\frac{-\eta_{\rm end} + i\gamma_{\rm end}\eta_{\rm end}}{\Delta\eta_{\rm end}}\right)\right] e^{ik\eta_{\rm r}}$$

Emergence of an exponential cut-off

Born approximation



Outline

- Primordial gravitational waves (PGWs): basics
- Calculation of the Power spectrum (PS) of PGWs
- Regulating the PGWs PS
- Smoothing the transitions
- The Born approximation
- Conclusions

Recap

- In order to have a correct behavior in the high frequency regime, the PS of PGWs must be regularized and the transitions (if any) must be smoothed out

- In a realistic model, any transition is infinitely smooth (not only linear) -> and this leads to an exponential cut-off in the high frequency regime.

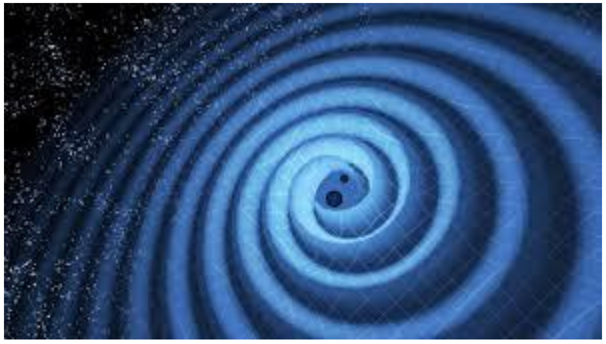
- The details of how the inflation to radiation transition proceeds will leave imprints of the PS -> interesting to probe reheating

- The same questions must be adressed for scalar PS and secondary GWs

Thank you for your attention!

$$\mathcal{P}_{\mathrm{T}}^{\mathrm{reg}}(k,\eta) = \mathcal{P}_{\mathrm{T}}(k,\eta) - \mathcal{P}_{\mathrm{T}}^{\mathrm{ad}}(k,\eta)$$

The regularized power spectrum is not positive definite; but the power spectrum is only a derived quantity:



$$\longrightarrow \left\langle 0 \left| \hat{h}_{mn}(\eta,\boldsymbol{x})\hat{h}_{ij}(\eta,\boldsymbol{x}') \right| 0 \right\rangle$$

Only one physical requirement: must be well-behaved for any (x,x')

$$\mathcal{P}_{\mathrm{T}}^{\mathrm{reg}}(k,\eta) = \frac{2k^2}{\pi}\int_0^{+\infty} \mathrm{d}r\, r\, \sin(kr)\delta^{im}\delta^{jn}\left[\left\langle 0 \left| \hat{h}_{mn}(\eta,\boldsymbol{x})\hat{h}_{ij}(\eta,\boldsymbol{x}') \right| 0 \right\rangle - \left\langle 0 \left| \hat{h}_{mn}(\eta,\boldsymbol{x})\hat{h}_{ij}(\eta,\boldsymbol{x}') \right| 0 \right\rangle^{\mathrm{ad}}\right]$$