REGULAR SPACETIMES: BLACK HOLES, BUBBLES AND WORMHOLES

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THE QUESTION

Many spacetimes (spherically symmetric, static or axisymmetric, stationary) in General Relativity or other theories of gravity, are known to have

'Horizons' AND 'Singularities'

Are there 'REGULAR' spacetimes in the same class, with or without horizons, which do NOT have singularities?

THE ANSWER

- No horizon, no singularity:

  Bubbles, Wormholes. ✓

- Horizon(s), no singularity:

  Regular black holes. ✓

- Horizon(s) and singularity:

  Black holes.

- No Horizon(s) but singular:

  Naked singularities.

EVENT HORIZON

- **Formal definition:**

$$B = M - J^{-}(\mathcal{I}^{+}) \quad ; \quad H = \partial B$$

M is the full manifold, B is the black hole region which is the set of points in M that are not in the causal past of future null infinity. H is the boundary of B called the horizon.

- **Working definition:**

For static, spherisymmetric, asymptotically flat metrics the roots of $g_{tt} = 0$ are the locations of the horizons (Vishveshwara (JMP, 1968)). These are also the infinite redshift surfaces and are null surfaces.

SINGULARITY

- **Curvature divergence:**

Location/region where atleast one of the

17 Zakhary–Mcintosh (GRG, 1997) invariants

diverge. This defines a curvature singularity.

Originally, 14 Narlikar–Karmarkar (PIAS-A,1949) invariants. Later, 16 Carminati-McLenaghan (JMP, 1991) invariants.

For an explicit listing of the invariants for Kerr-Newman type geometries, see, Overduin, Coplan, Wilcomb and Henry, Universe (2020).

- **Geodesic incompleteness:**

Follow observers along inextendible geodesics, i.e. geodesics whose domain of definition is the largest possible.

If the affine parameter of this geodesic does not cover the entire real line, the geodesic is said to be incomplete.

It means that the trajectory of our observer ends after or before a finite interval of proper time. Such a spacetime is singular in the sense of geodesic incompleteness.

Penrose (1965) first introduced geodesic incompleteness and its relation to singular spacetimes.

REGULAR BLACK HOLES

Horizons, no singularities

REGULAR BLACK HOLES

- Have horizon(s) but no curvature singularities.

- $r \to 0$ behaviour de Sitter, Minkowski, but never with divergent curvature scalars.

- Usually asymptotically flat. But may be asymptotically dS or AdS.

- Geodesically complete. Evades singularity theorems.

HISTORY

• Sakharov (1966), Gliner (1966): idea about de Sitter core.

• Bardeen (1968): first concrete metric example within GR.

• Roman and Bergmann (1983): stellar collapse without a singularity.

• Many other examples later: Dymnikova, Hayward, Neves-Saa, Frolov-Markov-Mukhanov, Mars et al....., Gurses-Gursey (rotating).

• Sources: nonlinear electrodynamics (Ayon-Beato, Garcia), scalar fields (Bronnikov-Fabris).

• Recent example: Simpson-Visser (2019)

BARDEEN SPACETIME

James Bardeen (1939-2022), GR5 (Tbilisi, 1968)

- **Line element:**

$$ds^2 = -\left(1 - \frac{b(r)}{r}\right) dt^2 + \frac{dr^2}{1-\frac{b(r)}{r}} + r^2 d\Omega_2^2$$

$$b(r) = \frac{2Mr^3}{(r^2+g^2)^{\frac{3}{2}}}$$

$\rightarrow g = 0$ **is the Schwarzschild. Asymptotically flat.**

- **Metric around** $r \rightarrow 0$**:**

$$-g_{00} = g^{11} = 1 - \frac{2M}{g^3} r^2 \left(1 - \frac{3}{2}\frac{r^2}{g^2} + ...\right)$$

$$\sim 1 - \frac{2M}{g^3} r^2$$ (**de Sitter**)

- **de Sitter core replaces singularity.**

- **Matter stress energy**

$$\rho = -\tau = \frac{1}{8\pi G}\frac{6Mg^2}{\left(r^2+g^2\right)^{\frac{5}{2}}}$$
$$p = \frac{1}{8\pi G}\frac{9Mr^2g^2-6Mg^4}{\left(r^2+g^2\right)^{\frac{7}{2}}}$$

- **At** $r = 0$, $\rho = -\tau = -p = \frac{1}{8\pi G}\frac{6M}{g^3}$, **behaves like a cosmological constant.**

- **Note that WEC, NEC hold.** $\rho > 0$, $\rho + \tau = 0$ **and** $\rho + p > 0$.

- **But SEC (NEC and** $\rho + \tau + 2p \geq 0$**) does not hold in the region** $r^2 < \frac{2}{3}g^2$ **which includes** $r = 0$.

- **Horizons:**

$-g_{tt} = 0$ **leads to the cubic:**

$$z^3 + \left(3\alpha^2 - 4\right) z^2 + 3\alpha^4 z + \alpha^6 = 0$$

where $z = \frac{r^2}{M^2}$**,** $\alpha = \frac{g}{M}$**.**

Discriminant $\Delta = 4\alpha^6 \left(\alpha^2 - \frac{16}{27}\right)$

$\rightarrow$ $\Delta > 0$, $\alpha^2 > \frac{16}{27}$, **single real root always negative.**

$\rightarrow$ $\Delta = 0$, $\alpha^2 = \frac{16}{27}$, **two real roots merge and positive, other always negative. One horizon.**

$\rightarrow$ $\Delta < 0$, $\alpha^2 < \frac{16}{27}$, **two real roots positive, third always negative. Two horizons.**

- Curvature singularities

$\rightarrow$ Ricci scalar:

$$R = \frac{24Mg^4 - 6Mg^2r^2}{(r^2+g^2)^{\frac{7}{2}}}$$

Positive at $r = 0$, approaches zero as $r \rightarrow \infty$. Never divergent.

$\rightarrow$ Kretschmann scalar:

$$K = \frac{12M^2\left(-4g^6r^2+47g^4r^4-12g^2r^6+8g^8+4r^8\right)}{(r^2+g^2)^7}$$

Positive at $r = 0$, approaches zero as $r \rightarrow \infty$. Never divergent.

All other scalars also finite. No curvature singularity. Geodesically complete (see Zhou, Modesto, PRD 2023).

SINGULARITY AVOIDANCE: HOW?

• Penrose singularity theorem (1965):

If the space-time contains a non-compact Cauchy hypersurface Σ and a closed future-trapped surface, and if the convergence condition holds for null u^i, then there are future incomplete null geodesics.

• Hawking-Penrose (1970):

If the convergence and generic conditions hold for causal vectors, there are no closed timelike curves and there exists at least one of the following: (i) a closed achronal imbedded hypersurface, (ii) a closed trapped surface, (iii) a point with re-converging light cone then the space-time has incomplete causal geodesics.

$\rightarrow$ In regular black holes there exist compact Cauchy surfaces (near the de Sitter core).

$\rightarrow$ The Strong Energy Condition is violated.

These help in evading the singularity theorems.

- **Borde's theorem:**

Arvind Borde (PRD 1997) showed that the existence of regular BH implies a topology change from $R \times S^2$ to S^3 as one approaches the de Sitter core.

NEVES-SAA

Neves and Saa, PLB 2014

- **Line element:**

$$ds^2 = -\left(1 - \frac{b(r)}{r}\right) dt^2 + \frac{dr^2}{1-\frac{b(r)}{r}} + r^2 d\Omega_2^2$$

$$b(r) = \frac{2Mr^p}{(r^q+g^q)^{\frac{p}{q}}};\ p, q > 0.$$

→ $p = 3$, $q = 2$ **is the Bardeen case.**

→ **Only for** $p = 3$ **one gets a core which is de Sitter.**

→ **For** $p \leq 3$, **WEC, NEC hold everywhere and SEC holds over a restricted region. For** $p > 3$ **WEC, NEC, SEC hold only in a restricted region.**

HAYWARD

Hayward, PRL 2006

- **Line element:**

$$ds^2 = -\left(1 - \frac{b(r)}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\Omega_2^2$$

$$b(r) = \frac{2Mr^3}{r^3 + g^3}$$

$\rightarrow$ **This is the $p = q = 3$ case in the Neves-Saa generalisation.**

- **WEC, NEC hold, SEC holds in a restricted region.**

- **No curvature singularity, de Sitter core but geodesically incomplete.**

SOURCES: NEVES–SAA

- Ayon-Beato, Garcia (1998-99) realised that sources for the Bardeen class could be

Nonlinear Electrodynamics

Generalisation by Fan and Wang (2016).

- Metric function $b(r)$:

$$b(r) = \frac{2}{\alpha}\frac{g^3 r^p}{(r^q+g^q)^{\frac{p}{q}}}$$

- Matter Lagrangian ($\mathcal{F} = F_{ij}F^{ij}$):

$$\mathcal{L} = \frac{4p}{\alpha}\frac{(\alpha\mathcal{F})^{\frac{q+3}{4}}}{\left(1+(\alpha\mathcal{F})^{\frac{q}{4}}\right)^{\frac{q+p}{q}}}$$

- Problem with such Lagrangians–fractional powers of $\mathcal{F}$–no purely electric solution.

DYMNIKOVA

Dymnikova, GRG 1992

- **Line element:**

$$ds^2 = -\left(1 - \frac{b(r)}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\Omega_2^2$$

$$b(r) = 2m\left(1 - \exp(-\frac{r^3}{r_*^3})\right) \text{ ; } r_*^3 = 2mr_0^2$$

- **For r very large $b(r) = 2m$. For small r, doing an expansion one gets $b(r) = \frac{r^3}{r_0^2}$ (de Sitter).**
- **NEC, WEC holds for the matter required. SEC violated for $r^3 < \frac{2r_*^3}{3}$.**
- **Horizons at the roots of $x = 1 - e^{-\frac{x^3}{y^2}}$, where $x = \frac{r_H}{2m}$, $y = \frac{r_0}{2m}$.**

General Relativity and Gravitation, Vol. 24, No. 3, 1992

Vacuum Nonsingular Black Hole†

Irina Dymnikova[1]

The spherically symmetric vacuum stress-energy tensor with one assumption concerning its specific form generates the exact analytic solution of the Einstein equations which for large r coincides with the Schwarzschild solution, for small r behaves like the de Sitter solution and describes a spherically symmetric black hole singularity free everywhere.

A tiny fish is better than a big cockroach
– Russian proverb

The year 1917 went down in history not only as the year when Lenin seized power in Russia to put the Marxist doctrine into practice but also as the year when de Sitter published his cosmological solution [1]

$$ds^2 = \left(1 - \frac{r^2}{r_0^2}\right) c^2 dt^2 - \frac{dr^2}{1 - (r^2/r_0^2)} - r^2(d\theta^2 + sin^2\theta d\varphi^2), \qquad (1)$$

where $r_0^2 = 3/\Lambda$, with the cosmological constant Λ responsible for the geometry. During several decades the physical essence of this solution remained obscure. In modern physics it has been mainly used as a simple testing ground for developing the quantum field techniques in curved space-time.

Fifty years later, it was understood [2–4] that the de Sitter geometry is generated by a vacuum with nonzero energy density $\varepsilon = \Lambda c^4/8\pi G$,

† This essay received the fifth award from the Gravity Research Foundation, 1991
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BRONNIKOV-FABRIS

Bronnikov, Fabris, PRL(2006)

- **Line element:**

$$ds^2 = -\left(1 - \frac{b(r)}{r}\right) dt^2 + \frac{dr^2}{1-\frac{b(r)}{r}} + r^2 d\Omega_2^2$$

$$b(r) = C\frac{r^3}{b^2} - \rho_0 \frac{r^3}{b^3}\left(\frac{b\sqrt{r^2-b^2}}{r^2} + \tan^{-1}\frac{\sqrt{r^2-b^2}}{b}\right)$$

→ **Exact RBH solution of Einstein gravity coupled to a phantom scalar and a potential $V(\phi, b, C)$.**

→ **Can be reduced to the Bronnikov–Ellis wormhole for a choice of parameters and coordinates.**

SIMPSON– VISSER

Simpson, Visser, JCAP 2019

- Line element:

$$ds^2 = -\left(1 - \frac{b(r)}{r}\right) dt^2 + \frac{dr^2}{A(r)\left(1 - \frac{b(r)}{r}\right)} + r^2 d\Omega_2^2$$

$$b(r) = 2m, \; A(r) = 1 - \frac{b_0^2}{r^2}$$

- Parameters b_0, m. If $b_0 < 2m$ the spacetime is a regular black hole. Horizon at $2m$. Throat at b_0.
- No curvature singularity. No proper source known. No de Sitter core.
- If $b_0 > 2m$, we can have a wormhole. $b_0 = 2m$, null throat–one way wormhole.

SOME OTHER MODELS

• Another, simpler metric with

$$b(r) = \frac{2Mr^3}{(r+g)^3}$$

Fan, Wang (PRD (2016)), Cadoni et al (PRD(2023)).

• LQG (Loop) corrected regular black holes Modesto, IJTP (2011).

• Pure Lovelock regular black holes Estrada, Aros (2024).

• Rotating regular black holes.

See Bambi (2013), H. Maeda (JHEP (2022)) for summary.

• **Models (generalised Hayward) with metrics not in the Schwarzschild gauge. Construction via Damour-Solodukhin trick (functionally same $-g_{tt}$, g^{rr} but with different parameters).** $x = \frac{r}{M}$**.**

$$-g_{tt} = 1 - \frac{2\sigma x^2}{x^3 + 2\sigma\kappa^2}\ ;\ g^{rr} = 1 - \frac{2x^2}{x^3 + 2\kappa^2}$$

PDR, SK (PRD(2022)).

• **Models with a singular (g=0) limit different from Schwarzschild (it is the $m = 0$ Einstein-Rosen bridge).**

$$-g_{tt} = g^{rr} = 1 - \frac{b_0^2 r^2}{\left(g^2 + r^2\right)^2}.$$

AK & SK, GRG (2024).

PROBLEMS, NEW PHYSICS?

- **Regularity versus the first law of black hole mech.** ($\delta M = \frac{\kappa}{8\pi}\delta A + ..$)?

Choosing $-g_{tt} = g^{rr} = f(r)$,

$$f(r) = 1 - \frac{2M}{r}\sigma(M, r, \alpha)$$

one can show, using two definitions of surface gravity:

$$\kappa = \frac{f'(r_H)}{2};\ \tilde{\kappa}^{-1} = r_H \frac{\partial r_H}{\partial M}$$

Thus, $\frac{\kappa}{\tilde{\kappa}} = \sigma + M\frac{\partial \sigma}{\partial M}$. If $\frac{\kappa}{\tilde{\kappa}} = 1$ (1st law) then, $f(r) = 1 - \frac{2M}{r} - \frac{2\zeta(r,\alpha)}{r}$

$\rightarrow$ a singular metric.

$\rightarrow$ The entropy $S = \frac{A}{4} + \delta S$, if we have a regular BH.

See Lan, Miao, Yang (NPB 2021); Murk and Soranidis (PRD 2023).

- **Cauchy horizon:**

$$H^{+}(S) = \overline{D^{+}(S)} - I^{-}(D^{+}(S)).$$

where S is a closed achronal set (Cauchy surface). $D^{+}(S)$ is future domain of dependence. Similar definition for $H^{-}(S)$.

Most regular black holes have an inner Cauchy horizon as in Reissner–Nordstrom. Thus, like RN spacetime, one has the issues of:

$\rightarrow$ **Loss of predictability**

$\rightarrow$ **Cauchy horizon instability**

Loss of predictability:

$\rightarrow$ Take an event P anywhere in the future of the Cauchy horizon.

$\rightarrow$ Conditions at P not uniquely determined by initial data on a spacelike hypersurface S outside the BH.

$\rightarrow$ The Cauchy horizon is the boundary of the region $D(S)$ for which the evolution is unique.

$\rightarrow$ Cauchy problem of GR well-posed in $D(S)$. At the Cauchy horizon the evolution ceases to be uniquely determined by initial data. This is the loss of predictability at the Cauchy horizon.

Figure from Poisson's thesis (1989): 'P' linked to Σ, but not 'Q'.

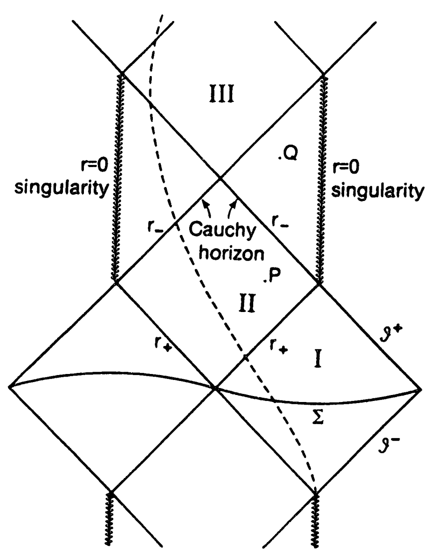


Figure 3.1: The Reissner-Nordstrøm spacetime. Region I is the exterior region of the black hole, belonging to "our" universe. Region II is the interior of the hole, and region III is another universe. If the black hole is the result of the gravitational collapse of a charged star, only the part of the diagram at the right of the dashed line is realized. The part at the left is then replaced by the interior of the collapsing star. More details are given in the text of section 3.2a).

Cauchy horizon instability:

A timelike observer moving towards the inner Cauchy horizon would 'see' the energy density of ingoing 'null dust' varying as $\frac{\mathbf{e}^{2\kappa_{-}\mathbf{v}}}{\mathbf{v}^{\gamma}}$ ($\gamma \geq 2$ (Price, 1972)). As $v \to \infty$ (inner Cauchy horizon) there will be an exponential divergence.

See the simple calculation in Brown, Mann and Modesto (PRD 2011) or Poisson's book (Pg 218) or Poisson's Phd thesis (Alberta, 1989)

- Recent debate on stability of the Cauchy horizon in the context of regular black holes. Elaborated in:

Bonano, Khosravi and Saueressig, PRD (2021).
Claim: Full backreaction accounted for. Stable cores.

Carballo-Rubio, Di Filippo, Liberati, Pacilio, Visser, JHEP (2021)
Claim: Exponentially growing Misner-Sharp mass at the inner horizon. Unstable.

Carballo-Rubio, Di Filippo, Liberati, Pacilio, Visser, JHEP (2022)

Claim: No mass inflation if κ_- vanishes. Find appropriate metrics.

Khodadi, Firouzjaee, PLB (2024)

$\kappa_- = 0$ spacetimes classically stable but quantum mechanically not so ($\langle T_{uu} \rangle$, $\langle T_{vv} \rangle$ w.r.t Hartle-Hawking or Unruh states, diverge at the inner horizon).

Bonano, Khosravi and Saueressig, PRD (2023).

Claim: Dynamical study including mass loss due to Hawking radiation. Stable cores.

- Problem both ways!!

Loss of predictability if Cauchy horizon is stable.

Divergent energy density $\rightarrow$ unstable CH $\rightarrow$ singularity formation.

• Gravitational collapse into RBH?

→ Oppenheimer–Snyder like collapse. No singularity but horizons form.

→ Interior flat FRW, exterior regular BH. Evolving boundary eqn.

→ Requires polytropic e.o.s., late time SEC violation and infinite time for collapse.

See Shojai, Sadeghi and Hassannejad (CQG 2023)

See earlier paper by Zhang et al (EPJC 2015) for a treatment using quantum gravity inspired models.

Review in D. Malafarina, 2209.11406

BUBBLES, WORMHOLES

No horizons, no singularities

• Usually, bubbles are dynamic, vacuum, nonsingular spacetimes.

• Wormholes are static, spherically symmetric, non-vacuum, nonsingular spacetimes.

• Usually both types exist more comfortably in theories beyond GR!

• Their existence within GR leads to problems on various fronts: eg. energy conditions (for wormholes).

• We will discuss just one example which will cover both types.

THE WITTEN BUBBLE

- **5D Schwarzschild:**

$$ds^2 = -\left(1 - \frac{b_0^2}{r^2}\right) d\tau^2 + \frac{dr^2}{1 - \frac{b_0^2}{r^2}}$$

$$+r^2 \left(d\eta^2 + \sin^2 \eta \, d\theta^2 + \sin^2 \eta \sin^2 \theta d\phi^2\right)$$

- **Double Wick rotation:**

$$\tau \to iR\chi, \; \eta \to -i\alpha t + \tfrac{\pi}{2}$$

$$ds^2 = -\alpha^2 r^2 dt^2 + \frac{dr^2}{1 - \frac{b_0^2}{r^2}} + r^2 \cosh^2 \alpha t \, d\theta^2$$

$$+r^2 \cosh^2 \alpha t \, \sin^2 \theta d\phi^2 + R^2 \left(1 - \frac{b_0^2}{r^2}\right) d\chi^2$$

- **Bubble of nothing**

(**E. Witten, NPB 1982**).

- **Vacuum solution in KK theory.**

- **Instability of the KK vacuum.**

- **Time-dependent solution.**

- **Ever expanding bubble.**

$\rightarrow$ **Note** g_{00}

$\rightarrow$ **Warped extra dimension** χ**.**

$\rightarrow$ **The expanding in time** S^2

NON-VACUUM WITTEN BUBBLE?

- **The vacuum Witten-bubble is:**

$$ds^2 = -\alpha^2 r^2 dt^2 + \frac{dr^2}{1 - \frac{b_0^2}{r^2}} + r^2 \cosh^2 \alpha t \, d\Omega_2^2 + R^2 \left(1 - \frac{b_0^2}{r^2}\right) d\chi^2$$

- **We now consider the following:**

$$ds^2 = -\alpha^2 r^2 dt^2 + \frac{dr^2}{1 - \frac{b_0^2}{r^2}} + r^2 \cosh^2 \rho_1 t \, d\Omega_2^2 + R^2 \left(1 - \frac{b_0^2}{r^2}\right) d\chi^2$$

- **This slight change ($\alpha \neq \rho_1$) allows us to put $\rho_1 = 0$ and get a static metric! But it is non-vacuum.**

- **Both non-static ($\rho_1 \neq \alpha \neq 0$), static ($\rho_1 = 0$, $\alpha \neq 0$) versions require matter satisfying the WEC, NEC!**
- **The static metric ($\rho_1 = 0$) is a wormhole.**

$$ds^2 = -\alpha^2 r^2 dt^2 + \frac{dr^2}{1 - \frac{b_0^2}{r^2}} + r^2 d\Omega_2^2 + R^2 \left(1 - \frac{b_0^2}{r^2}\right) d\chi^2$$

- **The $r = b_0$ section has a degenerate induced metric.**
- **The extra dimension disappears near the throat. It attains a constant (R^2) value at infinity.**

FURTHER GENERALISATION

- **Propose a class of metrics:**

$$ds^2 = -e^{2\psi(r)}dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\Omega^2_{D-3}$$
$$+R^2\left(1 - \frac{b(r)}{r}\right) d\chi^2$$

- **Ultrastatic case:** $\psi = 0$. **5D vacuum Einstein equation. Exact solution.**

$$b(r) = \frac{b_0^{D-4}}{r^{D-5}}$$

- **5D**

$$ds^2 = -dt^2 + \frac{dr^2}{1 - \frac{b_0}{r}} + r^2 d\Omega_2^2 + R^2 \left(1 - \frac{b_0}{r}\right) d\chi^2$$

Mentioned by M. Roberts in 2009 in an unpublished preprint. Also Bah and Heidmann (PRL 2021).

- **6D**

$$ds^2 = -dt^2 + \frac{dr^2}{1 - \frac{b_0^2}{r^2}} + r^2 d\Omega_3^2 + R^2 \left(1 - \frac{b_0^2}{r^2}\right) d\chi^2$$

χ **=constant section, Bronnikov–Ellis with** S^3**.**

• Further generalisations possible with different $b(r)$ and with matter satisfying or violating the energy conditions.

• The extra dimension decays from a finite value at infinity to a zero value at the throat.

• No issue of energy condition violation for vacuum. But we need higher dimensions.

More details in SK (GRG (2022), SK(PRD (2022)).

WORMHOLES IN 4D

(Morris, Thorne AJP & Morris, Thorne, Yurtsever PRL 1988)

- A Lorentzian line element ($r \geq b_0$):

$$ds^2 = -e^{2\psi(r)}dt^2 + \frac{dr^2}{1-\frac{b(r)}{r}} + r^2 d\Omega^2.$$

- $\psi(r)$ and $b(r)$ satisfy:

$\rightarrow$ $e^{2\psi(r)}$ has no zeros, is finite $\Rightarrow$ no horizons.

$\rightarrow$ $b(r = b_0) = b_0$, $\frac{b(r)}{r} \leq 1$ (Lorentzian signature) $\Rightarrow$ shape function $b(r)$ dictates shape of wormhole.

$\rightarrow$ $r \rightarrow \infty, \frac{b(r)}{r} \rightarrow 0$ $\Rightarrow$ asymptotically flat.

$\rightarrow$ No singularities.

• If the above hold for $\psi(r)$, $b(r)$ then we have a Lorentzian wormhole spacetime.

• Matter required for wormhole existence violates the Energy Conditions. (NEC: $\rho + \tau \geq 0$, $\rho + p \geq 0$).

• How to circumvent energy condition violations? Warped extra dimension? Modified theory of gravity? Localised violations?

• Localised violation appears to happen for a 4D spacetime which is a 4D section of the 5D spacetimes discussed earlier.

- **The χ = constant 4D spacetime with $e^{2\psi} = \alpha^2 r^2$, $b(r) = \frac{b_0^2}{r}$ yields a BE spacetime with localised NEC violations (see SK (PRD(2022)).**

- **Take the static metric:**

$$ds^2 = -\alpha^2 r^2 dt^2 + \frac{dr^2}{1 - \frac{b_0^2}{r^2}} + r^2 d\Omega_2^2 + R^2 \left(1 - \frac{b_0^2}{r^2}\right) d\chi^2$$

Consider the χ =constant section.

LOCALISING VIOLATIONS

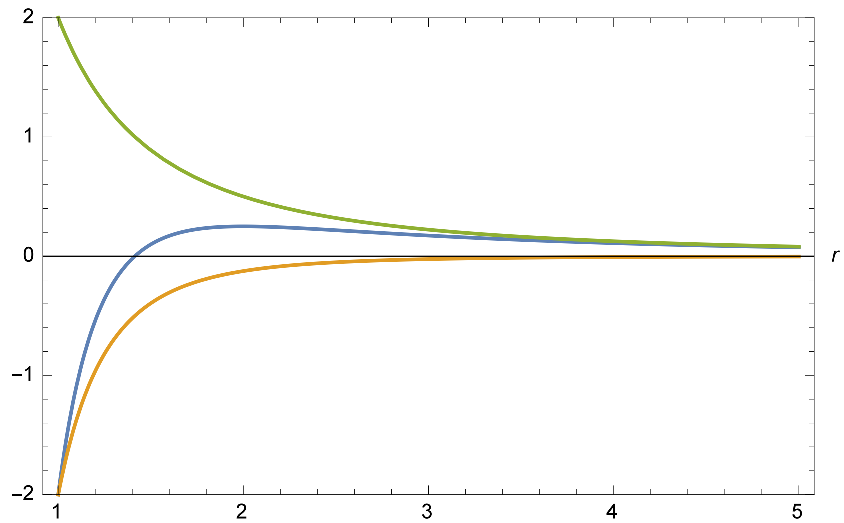
- **Variant** of BE (SK, PRD 2022)

$$ds^2 = -\alpha^2 r^2 dt^2 + \frac{dr^2}{1 - \frac{b_0^2}{r^2}} + r^2 d\Omega_2^2$$

- Not spacetime asymptotically flat. Only spatially asymptotically flat.
- Source $\rightarrow$ phantom scalar + NEC satisfying matter.
- $g_{00} = -\alpha^2 r^2$ can be found assuming one has a conformal Killing vector (Boehmer et al PRD 2007).
- Localises violation of NEC around the throat. ANEC violation can be controlled using parameters b_0, α.

$\rho + \tau$ versus r plot

$$ds^2 = -\alpha^2 r^2 dt^2 + \frac{dr^2}{1 - \frac{b_0^2}{r^2}} + r^2 d\Omega_2^2$$



→ **Green**: $b_0 = 0$ **geometry**

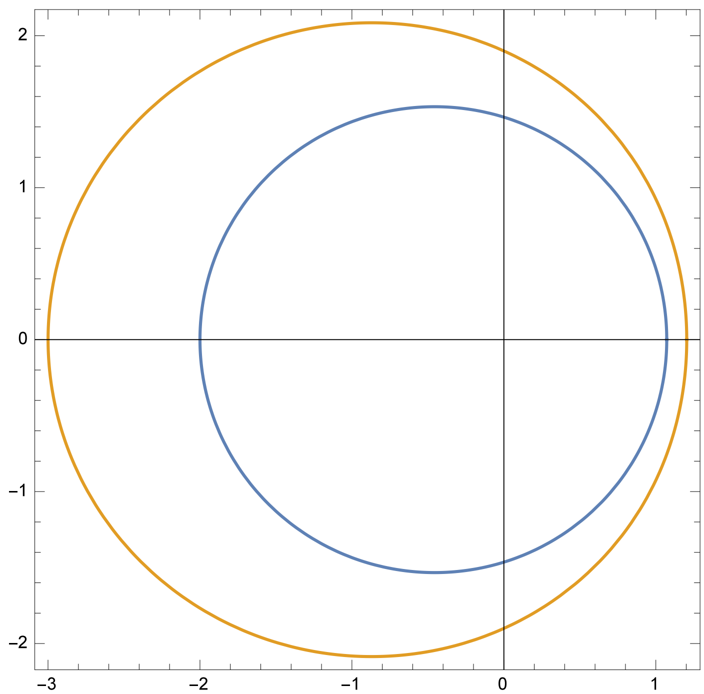
→ **Blue**: **Variant geometry**

→ **Yellow**: **BE geometry**

- $\rho + p \geq 0$ **always. True for general** $b(r)$ **instead of the** $b(r)$ **for BE.**

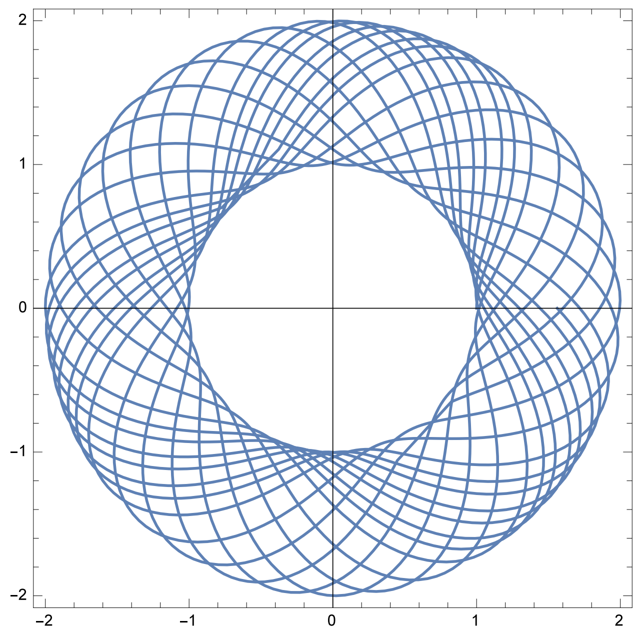
GEODESICS

- Exactly solvable timelike and null geodesics.
- Closed orbits



Different parameter sets for blue and yellow. Extra condition for closedness.

- **Open orbits**



Open but bounded between the throat b_0 and a $d_0 = \sqrt{\frac{E^2}{\alpha^2} - L^2} > b_0$

Timelike solutions in terms of Jacobian elliptic functions. Simple functions for null case. See SK, PRD 2022 for details.

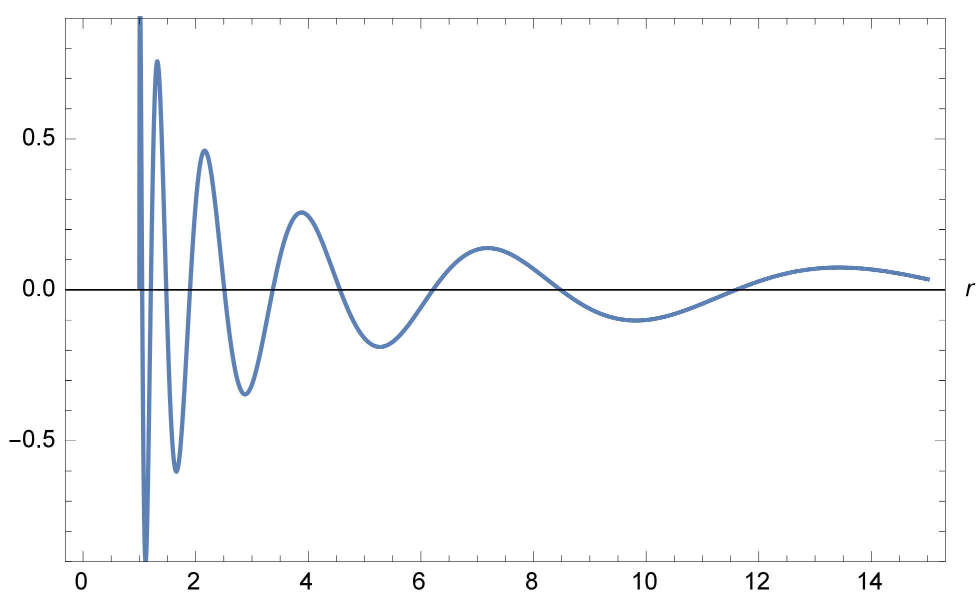
SCALAR WAVES

• The **massless scalar wave equation also exactly solvable** in this background.

$$\phi = e^{\pm i\omega t} R(r) Y(\theta, \phi).$$

$$R(r) = \frac{b_0}{r}\left(C_1 \sin\left[p \cosh^{-1}\frac{r}{b_0}\right] + C_2 \cos\left[p \cosh^{-1}\frac{r}{b_0}\right]\right)$$

where $p = \sqrt{\frac{\omega^2 - \alpha^2}{\alpha^2} + m(m+1)}$



Plot of the sine solution. **Decaying** with r.

SOME QUESTIONS

• Is it possible to construct a RBH spacetime (with rotation) with the following features:

$\rightarrow$ No curvature singularities, geodesically complete.

$\rightarrow$ Matter required satisfies energy conditions. A simple, viable model of matter exists.

$\rightarrow$ A single horizon: no inner Cauchy horizon.

- Is it possible to have binary systems with regular BHs? Can we have a merger remnant as a regular BH?

- Are there distinct features of QNMs and overtones which could characterise RBHs?

- What are the bounds on RBH parameters from EHT or say, S2 star data?

- Same questions can be asked for wormholes and bubbles too.

SOME REFERENCES

- Short review (RBH):

Lan, Yang, Guo and Miao
arxiv 2303.11696

- Book:

Regular Black Holes: Towards a New Paradigm of Gravitational Collapse (Ed. C. Bambi), Springer 2023

- Overview on wormholes:

Lobo and Rubiera-Garcia, Wormholes, energy conditions and time machines; and related articles in MG15 (Rome) Proceedings, World Scientific, 2022.