Quantum Ergodicity, Holography and Geometry

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Collaborators

- ArXiv: 2305.16028, 2310.13736 + work in progress

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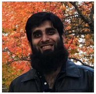


Premise: Holographic quantum field theory

- The prototypical example: AdS/CFT correspondence with Euclidean AdS metric $ds^2 = \frac{dx^2+dz^2}{z^2}$

$$Z_{grav}[\phi_0^i(x), \partial M] = \left\langle \exp\left(-\sum_i \int d^d x\, \phi_0^i(x)\mathcal{O}^i(x)\right)\right\rangle_{\partial M} \tag{1}$$

- The boundary conditions on the bulk fields

$$\phi_0^i(x) = \lim_{z\to 0} z^{\Delta-d}\phi(x,z) \tag{2}$$

- The mass of the bulk scalar field is related to the scaling dimension of the CFT operator by

$$m^2 = \Delta(d-\Delta) \tag{3}$$

Emergent spacetime ER=EPR

- Let us consider a thought experiment by Mark van Raamsdonk [2010]. Take two copies of a CFT on S^d. The Hilbert space is a tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. The quantum states are of the form

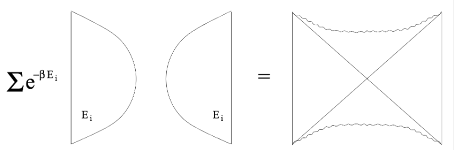
$$|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle \tag{4}$$

  If the CFT has a gravitational dual, then the geometry would be two disconnected spacetimes.

- Now consider a state in which the two copies are entangled

$$|\Psi_\beta\rangle = \sum_i e^{-\frac{1}{2}\beta E_i} |E_i\rangle |E_i\rangle \tag{5}$$

- This entangled state is dual to an eternal black hole with connected spacetime.



Local algebras in Quantum Field Theory

- Consider a quantum field theory defined on a spacetime manifold with coordinates t and local field $\phi(t)$.
- We consider local algebras of bounded operators that are closed under hermitian conjugation and contain the identity.

$$\tilde{\mathcal{A}} = \left\{ \sum \phi(f_1)\phi(f_2)\cdots \right\} , \qquad \phi(f) = \int dt\, f(t)\phi(t) \tag{6}$$

  where $f(t)$ is smooth function supported on some open set in the spacetime.
- Given a state, we close the algebra in the weak limit to get a von Neumann algebra. Or in other words we take the double commutant to get the von Neumann algebra $\mathcal{A} = (\tilde{\mathcal{A}})''$

Types of von Neumann algebras

Von Neumann algebras can be classified into 3 types:

- Type I_d: $d \times d$ matrix algebra (qudits), $\mathcal{B}(\mathcal{H}_d)$
- Type I_∞: Harmonic oscillator, $\mathcal{B}(\mathcal{H})$
- Type II_1
- Type II_∞: local algebras in quantum double models [Ogata 2022]
- Type III: local algebras in quantum field theories (eg. free scalar [Araki 1964])

Classical dynamical system

- A classical dynamical system is a triple (X, μ, T_t) where X is a measure space with measure μ and T_t is a dynamical flow parameterized by a discrete or continuous variable t. Let Σ be a σ-algebra of measurable sets in X.
- Example: Classical Phase Space
  $2n$-dimensional symplectic manifold with canonical symplectic 2-form

$$\omega = \sum_i \mathrm{d}q_i \wedge \mathrm{d}p_i \tag{7}$$

  and the canonical volume form ω^n.
- The Hamiltonian flow is induced by the Hamiltonian function $H(q_i, p_i)$ along with vector field V_H defined by the mapping $\mathrm{d}H = \omega(., V_H)$

$$V_H = \sum_i \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right) \tag{8}$$

- The Hamilton's equations of motion sets $V_H \sim \frac{\mathrm{d}}{\mathrm{d}t}$. This flow preserves H (energy) and the volume form ω^n. Hence, we will consider measure preserving flows that are invertible.

Classical ergodic hierarchy

- To keep track of the ergodic nature of the flow, we define the connected correlation function

$$C(A : B_t) = \mu(A \cap B_t) - \mu(A)\mu(B) \tag{9}$$

- A dynamical flow on (X, μ, T_t) is called ergodic if there is no subregion left invariant by the flow. In a a compact manifold, the long time average of an observable is equal to the measure space average. All connected correlation functions averaged over time decay to zero.

$$\forall A, B \in \Sigma : \qquad \lim_{T\to\infty} \frac{1}{T} \int_0^T dt\, C(A : B_t) = 0 \tag{10}$$

- Weak-mixing: If the average of the absolute values decay to zero

$$\forall A, B \in \Sigma : \qquad \lim_{T\to\infty} \frac{1}{T} \int_0^T dt\, |C(A : B_t)| = 0 \tag{11}$$

- Strong-mixing: If the correlation function decays to zero in the far future

$$\forall A, B \in \Sigma : \qquad \lim_{t\to\infty} C(A : B_t) = 0 \tag{12}$$

Kolmogorov and Anosov systems

- Kolmogorov-mixing: If the correlations with the entire past decay to zero

$$\forall A, \tilde{A} \in \Sigma : \qquad \lim_{t \to -\infty} \sup_{B \in B(t)} |C(A : B_t)| = 0 \tag{13}$$

  where $B(t)$ is the σ-algebra generated by $\{\tilde{A}_{t'} | t' < t\}$.

- Kolmogorov system: A classical dynamical system (X, μ, T_t) with measure-preserving flow is a classical K-system if there exists a σ-algebra of measurable sets $\Sigma_0 \subset \Sigma$ such that
  1. $\vee_{t \in \mathbb{R}} T_t \Sigma_0 = \Sigma$
  2. $\wedge_{t \in \mathbb{R}} T_t \Sigma_0 = \{\emptyset, X\}$
  3. $\forall t > 0,\ T_t \Sigma_0 \subset \Sigma_0$
- In K-systems, all connected correlators decay but the decay can be arbitrarily slow. It is an observed fact that in many relevant dynamical systems in physics, the correlators of observables decay in time.
- One can think of Anosov systems as a special class of Kolmogorov systems for which a dense set of correlators decay exponentially fast.

Examples of classical ergodic systems

- Rotations on a circle by irrational angle θ_0: ergodic

$$T^n(e^{i2\pi\theta}) = e^{i2\pi(\theta+n\theta_0)} \tag{14}$$

- Translations on a 2-torus $(0,1] \times (0,1]$ with irrational slope θ_0

$$T_t(x,y) = (x+t, y+\theta_0 t) \tag{15}$$

- Arnold's cat map: classical Anosov system

$$(x_{n+1}, y_{n+1}) = (x_n, y_n). \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \tag{16}$$

Rindler wedge as a classical Anosov system

- Consider the metric for $1+1d$ Minkowski spacetime $\mathbb{R}^{1,1}$

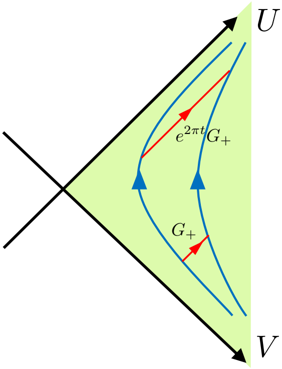
$$ds^2 = dx^+ dx^- \tag{17}$$

- The generators for null translations are $G_\pm = \partial_\pm$
- Consider a particle moving in a trajectory of constant acceleration $x^\pm(t) = x_0 e^{\pm t}$
- The dynamics is generated by the boost transformation e^{-tK}

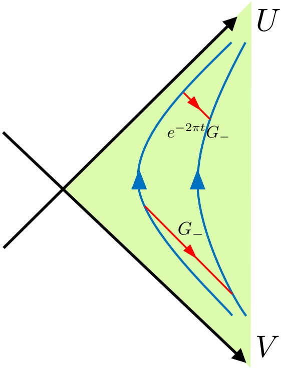
$$K = x^+\partial_+ - x^-\partial_- \tag{18}$$

- The null translations satisfy the algebra: $[K, G_\pm] = \mp G_\pm$
- The boost on the Rindler wedge is a K-system and along with the exponentially growing and decaying modes from the null translations, form a Anosov system

$$e^{-tK} e^{-sG_\pm} e^{tK} = e^{-se^{\pm t} G_\pm} \tag{19}$$



(a)



(b)

Quantum dynamical system

- The quantum generalization of a dynamical system is a triple $(\mathcal{A}, \omega, \tau_t)$ where $\mathcal{A}$ is a von Neumann algebra, ω a normal faithful state on $\mathcal{A}$ and the dynamical flow τ_t is a strongly continuous automorphism of $\mathcal{A}$.
- If the state ω satisfies the KMS condition with respect to τ_t, then τ_t describes modular dynamics

$$\omega(\tau_t(a)b) = \omega(b\tau_{t+i\beta}(a)) \tag{20}$$

- In the case of finite dimensional Hilbert space, an example of a KMS state is the thermal state with inverse temperature β and the modular flow is Hamiltonian time evolution.

Generalized Free Field (GFF) and holography

- A generalized free field (GFF) ϕ is a field for which is completely determined by its 2-point function. All higher 2n-point functions cluster and odd point functions vanish.
- This means that the Euclidean state of a GFF is a Gaussian state. [Duetsch, Rehren 2002]
- The 2-point function is determined by the spectral density $\rho(\omega)$

$$\langle \phi(t)\phi(t')\rangle = G(t-t') = \int_0^\infty d\omega\, \rho(\omega) e^{-i\omega(t-t')} \tag{21}$$

  Hence $G(\omega) = \rho(\omega)\Theta(\omega)$ where $\Theta(\omega)$ is the Heaviside step function.
- Holographic example: Single trace operators in the $N \to \infty$ limit of $\mathcal{N} = 4$ SYM above the Hawking-Page temperature form a type III_1 von Neumann algebra of GFF.

Quantum ergodic hierarchy

- In the quantum version, the measure in the connected correlation function is replaced by the expectation value in the state ω

$$f_{ab}^{conn}(t) = \omega(a\tau_t(b)) - \omega(a)\omega(b) \tag{22}$$

- Quantum ergodic: If all connected correlations averaged over time decays to zero

$$\forall a, b \in \mathcal{A} : \qquad \lim_{T\to\infty} \frac{1}{T}\int_0^T dt\, f_{ab}^{conn}(t) = 0 \tag{23}$$

- Quantum weak-mixing: If the absolute value of averaged connected correlator decays to zero

$$\forall a, b \in \mathcal{A} : \qquad \lim_{T\to\infty} \frac{1}{T}\int_0^T dt\, |f_{ab}^{conn}(t)| = 0 \tag{24}$$

- Quantum strong-mixing: If the connected correlator decays to zero in the far future

$$\forall a, b \in \mathcal{A} : \qquad \lim_{t\to\infty} f_{ab}^{conn}(t) = 0 \tag{25}$$

Discrete spectrum and almost periodicity

- Given a Hamiltonian H with energy eigenstates $|E_m\rangle$ and corresponding eigenvalues E_m, the thermofield double state (TFD) is defined on two copies of the Hilbert space $\mathcal{H}_L \otimes \mathcal{H}_R$ as

$$|1\rangle_\beta = \sum_m e^{-\frac{1}{2}\beta E_m} |E_m\rangle_L |E_m\rangle_R \tag{26}$$

- Expanded in the energy eigenbasis, we have

$$\begin{aligned} f_{ab}(t) &= {}_\beta\langle 1| e^{-itH_L} a^\dagger \otimes b^T e^{itH_L} |1\rangle_\beta \\ &= Z^{-1} \sum_{m,n} e^{-\frac{1}{2}\beta(E_m+E_n)} e^{i(E_m-E_n)t} a^*_{mn} b_{mn} \end{aligned} \tag{27}$$

- This is a B^2-Besicovitch almost periodic function of the form

$$f(t) = \sum_k \xi_k e^{i\lambda_k t} \tag{28}$$

with $\sum_k |\xi_k|^2 < \infty$.

Quantum Kolmogorov system

- Consider a von Neumann algebra $\mathcal{R}$ represented in a Hilbert space in the standard representation $\mathcal{H} = \mathcal{R}\,|\Omega\rangle$, and a strongly continuous unitary flow that preserves the vacuum state $e^{iHt}\,|\Omega\rangle = |\Omega\rangle$. We say this quantum dynamical system is a quantum K-system if it has a proper subalgebra $\mathcal{A} \subset \mathcal{R}$ such that
  1. Ergodicity: $(\vee_{s\in\mathbb{R}}\mathcal{A}_s)'' = \mathcal{R}$
  2. Strong-mixing: $\wedge_{s\in\mathbb{R}} = \lambda 1$
  3. Half-sided translations: For all $s > 0$ (or $s < 0$ but not both), we have $\mathcal{A}_s \subset \mathcal{A}$
- A subtlety that arises is that, as opposed to the classical case, quantum K-systems and quantum K-mixing, while intimately related, are not equivalent.
- The assumption of half-sided translation above is equivalent to the statement that $\mathcal{A}$ is a future algebra.

Quantum Anosov system: quantum fields in Rindler wedge

- A quantum Anosov system is defined as a quantum K-system with expanding and contracting unitary flows $e^{iG_\pm t}$ that preserve the state $G_\pm |\Omega\rangle = 0$

$$U(t)e^{isG_\pm} U^\dagger(t) = e^{ie^{\pm\lambda t} sG_\pm} \tag{29}$$

- Consider quantum field theory in $1+1d$ Minkowski spacetime $\mathbb{R}^{1,1}$ and the algebra of observables of the Rindler wedge.
- The generator of boost

$$K_W = \int_\Sigma d\Sigma^\mu B^\nu T_{\mu\nu} \tag{30}$$

with $B = i(x^+\partial_+ - x^-\partial_-)$ and the null translations $P_\pm$ form a quantum Anosov system

$$[P_+, P_-] = 0, \qquad [P_\pm, B] = \pm i P_\pm \ . \tag{31}$$

Deformation of the modular operator

- In the case of Rindler wedge, the modular flow is generated by the boost operator $\Delta^{it} = e^{-i2\pi t K_W}$.
- Consider a deformation of the Rindler wedge along the positive and negative null directions to get the regions $W(z^{\pm})$.
- The deformation in the modular operator is captured by

$$\pm G_{\pm} = \frac{1}{z^{\pm}}(K_W - K_{W(z^{\pm})}) \geq 0 \quad (32)$$

This is a positive operator.

- In the case of flat space Rindler wedge, we can explicitly evaluate them by choosing to integrate over the null plane $x^- = 0$ as the Cauchy surface

$$\begin{aligned} K_W &= i\int_{-\infty}^{\infty} dx^+ x^+ T_{++}(x^+, x^- = 0) \\ K_{W(z^+)} &= i\int_{-\infty}^{\infty} dx^+ (x^+ - z^+) T_{++}(x^+, x^- = 0) \\ G_+ &= i\int_{-\infty}^{\infty} dx^+ T_{++}(x^+, x^- = 0, x^i) = P_+ \ . \end{aligned} \quad (33)$$

Half-sided modular inclusions

- An important result of quantum ergodic theory is that for modular flow the ergodic hierarchy simplifies.
- In [Borchers 1992] it was shown for a von Neumann algebra in a standard GNS representation $\{\mathcal{H}, |\Omega\rangle, \mathcal{R}\}$, if we have an ergodic modular future subalgebra $\mathcal{A} \subset \mathcal{R}$, then the positive operator

$$G = K_{\mathcal{R}} - K_{\mathcal{A}} \geq 0 \tag{34}$$

generates a strongly continuous unitary flow $U(s) = e^{isG}$ corresponding to a growing Anosov mode with Lyapunov exponent $\lambda = 2\pi$

$$\forall s, t \in \mathbb{R} : \qquad \Delta_{\mathcal{R}}^{-it} e^{isG} \Delta_{\mathcal{R}}^{it} = e^{ie^{2\pi t} sG} \tag{35}$$

Emergence of Poincaré algebra

- Consider a pair of von Neumann subalgebras $\mathcal{A}^{\pm} \subset \mathcal{R}$ such that $\mathcal{A}^{+}$ and $\mathcal{A}^{-}$ are modular future and past subalgebras, respectively. Define

$$\begin{aligned} \mathcal{A}^{\pm}(s) &= \Delta_{\mathcal{R}}^{-is} \mathcal{A}^{\pm} \Delta_{\mathcal{R}}^{is} \\ \pm G_{\pm}(s) &= \mathcal{K}_{\mathcal{R}} - \mathcal{K}_{\mathcal{A}^{\pm}(s)} \end{aligned} \tag{36}$$

  and consider the algebra $\mathcal{A}^{+}(-s)$ and $\mathcal{A}^{-}(s)$ and correspondingly $G^{+}(-s)$ and $G^{-}(s)$ for $s \gg 1$. Then, in the large s scaling limit such that $z^{+} z^{-} e^{-4\pi s} \ll 1$, we have an emergent Poincaré algebra

$$\begin{aligned} \Delta_{\mathcal{R}}^{-it} e^{iz^{\pm} G_{\pm}} \Delta_{\mathcal{R}}^{it} &= e^{ie^{2\pi t} z^{\pm} G_{\pm}} \\ e^{iz^{+} G_{+}(-s)} e^{iz^{-} G_{-}(s)} e^{-iz^{+} G_{+}(-s)} &= e^{iz^{-} G_{-}(s) + O(z^{+} z^{-} e^{-4\pi s})} \end{aligned}$$

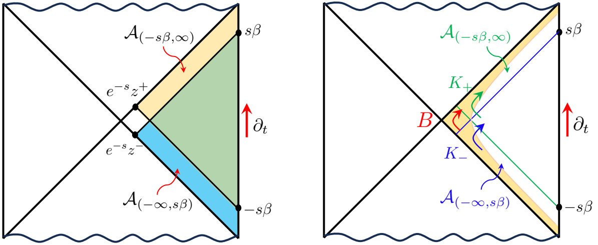
Curved spacetimes with bifurcate Killing horizons

- To give a bulk interpretation for the emergent Poincaré algebra, we consider spacetimes with a bifurcate Killing horizon with the metric

$$ds^2 = A(x^+x^-, x^i)dx^+dx^- + B(x^+x^-, x^i)dx^i dx_i \tag{37}$$

- It was shown in [Sewel 1982, Summers, Verch 1996] that in the vacuum representation of a QFT with the above metric, the action of the modular flow of W is the Killing flow generated by $B = i(x^+\partial_+ - x^-\partial_-)$.

Bulk interpretation of the emergent Poincaré algebra - near horizon limit



(a) (b)

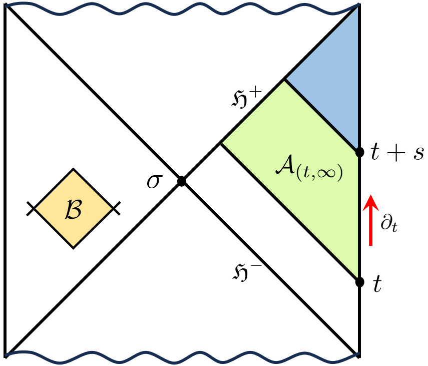
- (a) Near-horizon limit in the Hartle-Hawking state of an eternal AdS black hole dual to boundary GFF time interval algebras. (b) In the vicinity of the bifurcation surfaces, we obtain an emergent Poincaré group which has a corresponding analog in the boundary.

Second law in quantum K-systems

- In the thermodynamic limit, the expectation is that genuinely interacting quantum systems thermalize. In particular, we expect the emergence of a second law of thermodynamics that postulates the existence of a non-negative function of the state called entropy that grows monotonically in time.
- Intuitively, we can justify the emergence of the second law as follows: consider the subspace of all the observables that we can access from time t to eternity and denote it by $\mathcal{S}_{(t,\infty)}$. This is a future operator system. Forward time evolution is the restriction map on the future observables:

$$\forall s > 0 : \qquad e^{isH}\mathcal{S}_{(t,\infty)}e^{-isH} \subset \mathcal{S}_{(t,\infty)} \tag{38}$$

- We say time-evolution is a half-sided translation of the future operator system $\mathcal{S}_{(t,\infty)}$. Any information-theoretic measure that satisfies the data-processing inequality decreases monotonically over time. Multiplying any such measure by a minus sign, we obtain a monotonically increasing coarse-grained entropy; otherwise known as a second law of thermodynamics.


$\mathfrak{H}^+$
$t+s$
$\mathcal{A}_{(t,\infty)}$
σ
$\mathcal{B}$
∂_t
$\mathfrak{H}^-$
t

Second law from future subalgebras

- Future subalgebras $\mathcal{A}_{(t,\infty)}$ correspond to all observables we can measure from time t until eternity. They are very special, as when they exist, forward time evolution acts on them as the restriction map which is a unital completely positive (CP) map (the Heisenberg picture of a quantum channel).
- Consider the mutual information between the future algebra of the right boundary $\mathcal{A}_{(t,\infty)}$ and any subalgebra of the left boundary $\mathcal{B} \subset \mathcal{A}'_{(-\infty,\infty)}$ as an entropy function:

$$S(t) := I(\mathcal{A}_{(t,\infty)} : \mathcal{B}) \tag{39}$$

- It follows from strong subadditivity of entanglement entropy (the monotonicity of mutual information under partial trace) that

$$\forall t_1 \leq t_2 : \qquad S(t_1) = I(\mathcal{A}_{(t_1,\infty)} : \mathcal{B}) \geq I(\mathcal{A}_{(t_2,\infty)} : \mathcal{B}) = S(t_2) \tag{40}$$

Current work: beyond Poincaré algebra from the boundary

- Half-sided Modular intersections [Wiesbrock 1993]: Let $\mathcal{M}, \mathcal{N}, \mathcal{N} \cap \mathcal{M}$ be von Neumann algebras with a common cyclic and separating vector Ω. If $(\mathcal{M}, \mathcal{M} \cap \mathcal{N}, \Omega)$ and $(\mathcal{N}, \mathcal{M} \cap \mathcal{N}, \Omega)$ are $-$-half-sided, resp. $+$-half-sided modular inclusions such that $J_{\mathcal{N}} \mathcal{M} J_{\mathcal{N}} = \mathcal{M}$, then the modular unitaries $\Delta^{it}_{\mathcal{M}}$, $\Delta^{is}_{\mathcal{N}}$, $\Delta^{iu}_{\mathcal{N} \cap \mathcal{M}}$, $s, t, u \in \mathbb{R}$, generate a faithful continuous unitary representation of $PSL(2, \mathbb{R})$.
- Using $\pm$-modular intersections, Wiesbrock showed that you can canonically define a conformal field theory on S^1 only from the relative positions of the subalgebras.
- From the holographic perspective, AdS_2 metric has $PSL(2, \mathbb{R})$ as its isometry group. Our current effort is to work out the details of the conformal GFF theory on the boundary of AdS_2 and show the emergence of a geometric $PSL(2, \mathbb{R})$ algebra in the bulk.

THANK YOU!