# Spherical Collapse of Fuzzy Dark Matter

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Based on the work:

V. Sreenath, Phys. Rev. D **99**, 043540 (2019).

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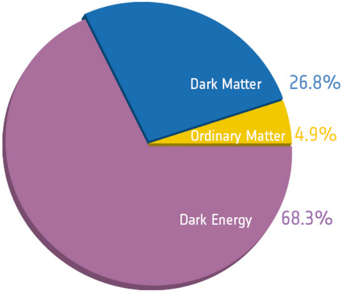
# Plan of the talk

1. Introduction
2. Fuzzy dark matter
3. Gross-Pitaevskii-Poisson system and equivalent fluid representation
4. Spherical collapse model
5. Virialization
6. Summary and discussion

# Introduction

# Standard model of cosmology

Standard model of cosmology, namely the $\Lambda$CDM model[1] has been a grand success. We have been able to model our universe using six basic parameters[2].




| Parameter | Best Fit |
|---|---|
| $\Omega_b h^2$ | $0.02233 \pm 0.00015$ |
| $\Omega_c h^2$ | $0.1198 \pm 0.0012$ |
| $100\,\theta_{\mathrm{MC}}$ | $1.04089 \pm 0.00031$ |
| $\tau$ | $0.0540 \pm 0.0074$ |
| $\ln(10^{10}\,A_s)$ | $3.043 \pm 0.014$ |
| $n_s$ | $0.9652 \pm 0.0042$ |

However, many questions remain!

[1] Figure from: http://planck.cf.ac.uk/results/cosmic-microwave-background.
[2] Planck Collaboration: N. Aghanim et al., arXiv: 1807.06209[astro-ph.CO].

# Small scale problems of dark matter

Despite the success of CDM at large scales, it is plagued with issues at small scales[3] ($< 10\mathrm{kpc}$). Of them, two most pertinent issues are:

- Core vs Cusp: CDM predicts that the halos have a cusp in the density profile at its centre. However, observations of low surface brightness galaxies and dwarf galaxies indicate that the density profiles at the centre of halos are shallower and hence has a core.
- Missing satellites: Simulations of CDM over predicts the number of dwarf satellites in local group by an order of magnitude.

Of these two issues, it has been suggested that the latter can be alleviated to an extent if one considers the effects of baryons[4].

[3] For a recent review, see James S. Bullock and Michael Boylan-Kolchin, Ann. Rev. Astron. Astrophys. **55**, 343387 (2017).
[4] S. Garrison-Kimmel *et al.* , arXiv:1806.04143 [astro-ph.GA].

# Some proposals for overcoming small scale issues

In order to overcome the small scale issues several alternatives to CDM has been proposed.

- Warm Dark Matter: In this model[5], dark matter particles possess a thermal velocity which allows them to free stream. This free streaming suppresses the formation of small scale structure and resolves the core-cusp problem.

- Collisional Dark Matter: In this model[6], dark matter particles interact with each other. The presence of collisions, provides a way to solve the issues at small scales.

In this work, we will consider a different approach to resolving the small scale issues.

[5] See, for instance, Y. P. Jing, Modern Physics Letters A **16**, 17951800 (2001).

[6] See, for instance, Paolo Salucci and Nicola Turini, arXiv:1707.01059 [astro-ph.CO].

# Fuzzy dark matter

## Fuzzy dark matter

- In Fuzzy Dark Matter (FDM)[7], the dark matter is composed of ultra light bosons of mass $m \simeq 10^{-24} - 10^{-22}\mathrm{eV}$, which exist as a Bose Einstein Condensate (BEC).

- All the large scale properties of the FDM are expected to be similar to that of CDM. However at small scales, the quantum properties of the BEC affects the formation of structure.

- Due to small mass of bosons, their de Broglie wavelength is of the order of $\mathrm{kpc}$ scales,

$$\lambda_{dB} = \frac{h}{p} = \frac{h}{m_b\, v_b} = 1.20 \times \left(\frac{10^{-22}\mathrm{eV}}{m_b}\right) \times \left(\frac{100\mathrm{km/s}}{v_b}\right) \mathrm{kpc}\ .$$

- The de Broglie wavelength manifests itself as a Jeans length below which the quantum pressure due to the uncertainty principle acts against gravity. Thus, below the de Broglie wavelength, the pressure suppresses the formation of structure and flattens the density profile.

[7] Wayne Hu, Rennan Barkana, and Andrei Gruzinov, Phys. Rev. Lett. **85**, 11581161 (2000).

# Axions

- A candidate for ultra light bosonic dark matter are axions[8]. Axions are spin zero periodic fields and they arise in several scenarios.

- Due to the periodicity, the axion field possess a quasi shift symmetry *i.e.* the shift symmetry is only partly broken which makes it nearly massless or ultra light.

- Axions are described by the action[9]

$$S = \int \mathrm{d}^4x \sqrt{-g} \left[ \frac{1}{2} F^2 g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - \mu^4 \left( 1 - \cos(\phi) \right) \right]$$

  where $\phi$ is the dimensionless field and mass of the field, $m = \mu^2/F$ where $\mu$ and $F$ are two parameters.

- The axion is governed by the equation of motion,

$$\ddot{\phi} + 3H\dot{\phi} + m^2 \sin(\phi) = 0.$$

[8] D. J. E. Marsh, Phys. Rept. 643, 179 (2016).
[9] Lam Hui, Jeremiah P. Ostriker, Scott Tremaine, and Edward Witten, Phys. Rev. D **95**, 043541 (2017).

# Axions as FDM candidate

- The axion is governed by the equation of motion,

$$\ddot{\phi} + 3\,H\,\dot{\phi} + m^2\,\sin(\phi) = 0.$$

- Early on in the universe, $H >> m^2$, in that limit the growing solution is $\phi \propto \mathrm{constant}$.

- As universe expands, $H$ becomes comparable to $m^2$. In that limit, the axion has an oscillating solution which decays as $\phi(t) \propto a(t)^{-3/2}$, *i.e.* the energy density of the axion field, $\rho_\phi \propto a^{-3}$.

- Thus, in this oscillatory phase, the axion behaves like classical CDM.

- By analyzing the perturbed Klein Gordon equation of the axions in the non-relativistic limit, one can rewrite the Klein-Gordon equation as a Gross-Pitaevskii equation[10]. Hence, one can interpret the axions as a BEC.

[10] A. Suarez and P.-H. Chavanis, Phys. Rev. D **92**, 023510 (2015).

# Perturbed Klein-Gordon Equation

Consider the following action for axions,

$$S = \int \frac{\mathrm{d}^4x}{\hbar\, c^2} \sqrt{-g} \left[ \frac{1}{2} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} - \frac{1}{2} \frac{m^2\, c^2}{\hbar^2} |\tilde{\phi}|^2 + \frac{\lambda}{4} |\tilde{\phi}|^4 \right]$$

In the non-relativistic limit, *i.e.* when $c \to \infty$, one can write the Klein-Gordon equation as

$$\tilde{\phi}'' + 2\mathcal{H}\, \tilde{\phi}' - \nabla^2 \tilde{\phi} = -\frac{m^2\, c^2}{\hbar^2} \tilde{\phi} + \frac{\lambda}{3!} |\phi|^2 \tilde{\phi} - 2\, \frac{m^2}{\hbar^2} \Phi \tilde{\phi}$$

where $\Phi$ is the gravitational potential and we have adopted the perturbed FLRW metric

$$\mathrm{d}s^2 = \left( 1 + \frac{2\, \Phi}{c^2} \right) c^2\, \mathrm{d}t^2 - a(t)^2 \left( 1 - \frac{2\, \Phi}{c^2} \right) \mathrm{d}\, x^2 .$$

Writing the scalar field in terms of a complex scalar field,

$$\tilde{\phi} = \sqrt{2}\, \Re \left( \psi(\eta,\, \vec{x})\, e^{-i\, \frac{m\, c^2}{\hbar} \int \mathrm{d}\eta' a(\eta)} \right)$$

one obtains the Gross-Pitaevskii equation which describes a Bose-Einstein condensate.

# Gross-Pitaevskii-Poisson system and equivalent fluid representation

# Gross-Pitaevskii-Poisson System

Since the axions exist as a BEC, the system of interest is a BEC evolving under the effect of gravity. The state of such a BEC is described by the condensate wave function $\psi(t, \overrightarrow{r})$ governed by the Gross-Pitaevskii-Poisson (GPP) system,

$$\begin{aligned} i\hbar\frac{\partial\psi(t, \overrightarrow{r})}{\partial t} &= -\frac{\hbar^2}{2\,m}\nabla^2\psi(t, \overrightarrow{r}) + m\,\Phi(t, \overrightarrow{r})\,\psi(t, \overrightarrow{r}) + \frac{4\,\pi\,a_s\,\hbar^2}{m^2}|\psi(t, \overrightarrow{r})|^2\psi(t, \overrightarrow{r}) \\ \nabla^2\Phi(t, \overrightarrow{r}) &= 4\,\pi\,\mathrm{G}\ |\psi(t, \overrightarrow{r})|^2, \end{aligned}$$

where,

$m$ is the mass of boson,

$\Phi(t, \overrightarrow{r})$ is the gravitational potential,

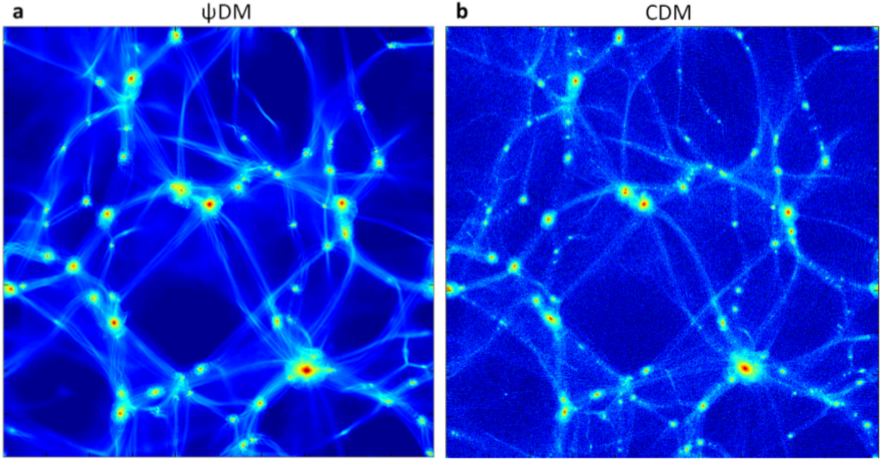
$\rho(t, \overrightarrow{r}) = |\psi(t, \overrightarrow{r})|^2$ is the mass density and

$a_s$ is the s-wave scattering length of bosons. A positive, zero and negative value of $a_s$ implies a repulsive, nil and attractive self-interaction of bosons respectively.

Structure formation in this system can be studied using numerical simulations.

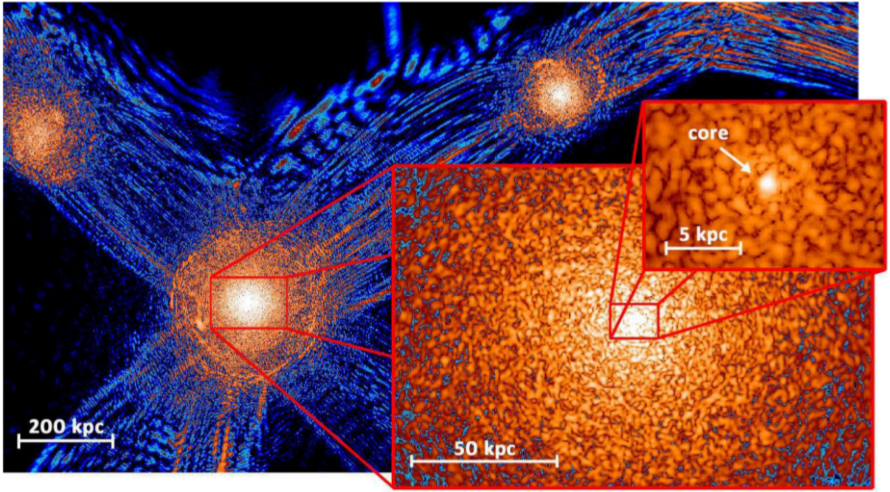
# Numerical simulations : Large scales

Numerical simulations[11] show that, at large scales, structures formed in FDM resembles that formed in CDM, as illustrated in this figure.




[11] Picture from Hsi-Yu Schive, Tzihong Chiueh, and Tom Broadhurst, Nature Phys. **10**, 496499 (2014).

A more resolved view[12] shows the differences from that of CDM.




[12] Picture from Hsi-Yu Schive, Tzihong Chiueh, and Tom Broadhurst, Nature Phys. **10**, 496499 (2014).

# Results from numerical simulations

- At large scales, simulations show that the structures formed in FDM is similar to that produced in CDM.
- High resolution simulations show that FDM halo centers have a solitonic core with outer profiles similar to the NFW profile.
- As the solitonic core accretes more matter, it grows and are surrounded by virialized halos with fine-scale, large-amplitude fringes.
- The surrounding halos are supported against gravity by quantum and turbulent pressure and hence fluctuates in density and velocity.

# Motivation

- Though the numerical simulations are required to have an exact understanding of structure formation, analytical approximations often provide useful insights.

- With this motivation, we will study the simplest model of nonlinear structure formation, namely the spherical collapse model for FDM.

# Gross-Pitaevskii-Poisson system

The system under consideration is that of a self-gravitating BEC which is governed by the Gross-Pitaevskii-Poisson (GPP) system,

$$
\begin{aligned}
i\hbar\frac{\partial\psi(t,\,\vec{r})}{\partial t} &= -\frac{\hbar^2}{2\,m}\nabla^2\psi(t,\,\vec{r}) + m\,\Phi(t,\,\vec{r})\,\psi(t,\,\vec{r}) + \frac{4\,\pi\,a_s\,\hbar^2}{m^2}|\psi(t,\,\vec{r})|^2\psi(t,\,\vec{r}) \\
\nabla^2\Phi(t,\,\vec{r}) &= 4\,\pi\,G\;|\psi(t,\,\vec{r})|^2,
\end{aligned}
$$

where,

$m$ is the mass of boson,

$\Phi(t,\,\vec{r})$ is the gravitational potential,

$\rho(t,\,\vec{r}) = |\psi(t,\,\vec{r})|^2$ is the mass density and

$a_s$ is the s-wave scattering length of bosons. A positive, zero and negative value of $a_s$ implies a repulsive, nil and attractive self-interaction of bosons respectively.

In order to study the evolution of a spherical shell of FDM, it is convenient to rewrite the GPP system as fluid equations.

# Madelung transformation

It is often convenient to express the GPP equations, describing the FDM halo, in terms of fluid variables, namely density and velocity[13]. This can be achieved by performing a Madelung transformation[14],

$$\psi(t, \overrightarrow{r}) = \sqrt{\rho(t, \overrightarrow{r})}\, \exp(i\, S(t, \overrightarrow{r})/\hbar)$$

where $\rho(t, \overrightarrow{r})$ and $S(t, \overrightarrow{r})$ are real quantities.

[13] P. H. Chavanis, A&A **537**, A127 (2012).

[14] E. Madelung, Zeitschrift für Physik **40**, 322326 (1927).

# Fluid equations in a static universe

Up on applying the transformation to the GPP system, defining

$$\vec{u}(t,\vec{r}) \equiv \frac{\vec{\nabla} S(t,\vec{r})}{m},$$

equating real and imaginary parts and using the identity,

$$(\vec{u}\cdot\vec{\nabla})\vec{u} = \vec{\nabla}(u^2/2) - \vec{u}\times(\vec{\nabla}\times\vec{u}) = \vec{\nabla}(u^2/2),$$

we obtain,

$$\begin{aligned}
\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) &= 0, \\
\frac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u} &= -\frac{\vec{\nabla}P}{\rho} - \vec{\nabla}\Phi - \frac{\vec{\nabla}Q}{m}, \\
\nabla^2\Phi &= 4\pi\,\mathrm{G}\;\rho,
\end{aligned}$$

which are respectively the continuity, Euler and Poisson equations of a fluid with density $\rho$ and velocity $\vec{u}$.

# Fluid equations in a static universe

Thus, using Madelung transformation, one can rewrite the GPP system as fluid equations, namely,

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) &= 0, \\
\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \vec{u} &= -\frac{\vec{\nabla} P}{\rho} - \vec{\nabla} \Phi - \frac{\vec{\nabla} Q}{m}, \\
\nabla^2 \Phi &= 4 \pi \mathrm{G} \; \rho.
\end{aligned}$$

Some remarks are in order,

- Since, $\vec{u}(t, \vec{r}) \equiv \frac{\vec{\nabla} S(t, \vec{r})}{m}$, we see that $\vec{u}$ is irrotational.
- In the Euler equation, the quantum pressure is given by,

$$Q(t, \vec{r}) = -\frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}} = -\frac{\hbar^2}{4m} \left[ \frac{\nabla^2 \rho}{\rho} - \frac{1}{2} \frac{(\nabla \rho)^2}{\rho^2} \right],$$

- The pressure arising from self-interactions is given by,

$$P(t, \vec{r}) = \frac{2 \pi a_s \hbar^2}{m^3} \rho^2.$$

Note that the above equation describes an equation of state of a polytrope of index one.

# Fluid equations in an expanding universe

In an expanding universe, $\vec{r}(t) = a(t)\,\vec{x}$.

Using the relation,

$$\left.\frac{\partial}{\partial t}\right|_{\vec{r}} = \left.\frac{\partial}{\partial t}\right|_{\vec{x}} - H\,\vec{x}\cdot\vec{\nabla},$$

where $H(t) = \dot{a}(t)/a(t)$ is the Hubble parameter, the fluid equations can be written as,

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} - H\,(\vec{x}\cdot\vec{\nabla})\,\rho + \frac{\vec{\nabla}\cdot(\rho\,\vec{u})}{a} = 0,\\
&\frac{\partial \vec{u}}{\partial t} - H\,(\vec{x}\cdot\vec{\nabla})\,\vec{u} + \frac{(\vec{u}\cdot\vec{\nabla})\,\vec{u}}{a} = -\frac{\vec{\nabla}P}{a\,\rho} - \frac{\vec{\nabla}\Phi}{a} - \frac{\vec{\nabla}Q}{a\,m},\\
&\nabla^2\Phi = 4\,\pi\,\mathrm{G}\;a^2\,\rho,
\end{aligned}$$

where $\vec{\nabla}$ is now with respect to $\vec{x}$.

# Perturbed fluid equations in an expanding universe

Let us now split the density, velocity and gravitational potential in to its background part and a perturbation on top of it, *i.e.*

$$\rho = \rho_b(1+\delta),\ \text{where}\, \delta = \delta\rho/\rho_b$$
$$\vec{u} = H\,\vec{r} + \vec{v},\ \text{and}$$
$$\Phi = \Phi_b + \Phi_p,\ \text{where}\, \Phi_b = -\ddot{a}\, r^2/(2\,a).$$

Using these definitions, one could write the perturbed part of the fluid equations as,

$$\frac{\partial\delta}{\partial t} + \frac{\vec{\nabla}}{a}\cdot\left[\vec{v}\,(1+\delta)\right] = 0$$
$$\frac{\partial\vec{v}}{\partial t} + H\,\vec{v} + \frac{(\vec{v}\cdot\vec{\nabla})\vec{v}}{a} = -\frac{4\,\pi\,a_s\,\hbar^2}{a\,m^3}\vec{\nabla}\rho - \frac{\vec{\nabla}\Phi_p}{a} + \frac{\hbar^2}{4\,m^2\,a^3}\vec{\nabla}\left[\frac{\nabla^2\rho}{\rho} - \frac{1}{2}\frac{(\nabla\rho)^2}{\rho^2}\right]$$
$$\nabla^2\Phi_p = 4\,\pi\,\mathrm{G}\;\, a^2\,\rho_b\delta.$$

In writing the perturbed part of Euler equation, we have retained the full density, $\rho$, on the right hand side for later convenience.

# Spherical collapse model

# Evolution of a spherical shell

Consider a spherically overdense distribution of FDM. Consider a spherical shell of radius $R(t) = a(t)X(t)$, enclosing certain mass, centered in the overdense region. A fluid element on that shell would have a velocity, $\vec{u} = H\,\vec{R} + \vec{v}$, where the velocity of the fluid element is in radial direction.

The acceleration of that fluid element can be computed as,

$$\frac{\mathrm{d}^2\vec{R}}{\mathrm{d}t^2} = \frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = \dot{H}\,\vec{R} + H\,(H\,\vec{R} + \vec{v}) + \frac{\partial\vec{v}}{\partial t} + \frac{(\vec{v}\cdot\vec{\nabla})}{a}\vec{v}.$$

# Equation of motion of a spherical shell

Up on using the perturbed Euler equation and the fact that $\vec{\nabla}\Phi_b = -\ddot{a}\,\vec{R}/a^2$, we obtain,

$$\frac{\mathrm{d}^2\vec{R}}{\mathrm{d}t^2} = -\frac{\vec{\nabla}\Phi_b}{a} - \frac{4\,\pi\,a_s\,\hbar^2}{a\,m^3}\vec{\nabla}\rho - \frac{\vec{\nabla}\Phi_p}{a} + \frac{\hbar^2}{4\,m^2\,a^3}\vec{\nabla}\left[\frac{\nabla^2\rho}{\rho} - \frac{1}{2}\frac{(\nabla\rho)^2}{\rho^2}\right].$$

Combining the background and perturbed parts of the gravitational potential, one can write the equation of motion of the spherical shell as,

$$\frac{\mathrm{d}^2\vec{R}}{\mathrm{d}t^2} = -\frac{4\,\pi\,a_s\,\hbar^2}{m^3}\vec{\nabla}\rho - \vec{\nabla}\Phi + \frac{\hbar^2}{4\,m^2}\vec{\nabla}\left[\frac{\nabla^2\rho}{\rho} - \frac{1}{2}\frac{(\nabla\rho)^2}{\rho^2}\right],$$

where the spatial derivatives are now with respect to $r$ and are evaluated on the shell, $r = R(t)$.

# Equation of motion of a spherical shell

The equation of motion of the spherical shell is,

$$\frac{\mathrm{d}^2 \overrightarrow{R}}{\mathrm{d}t^2} = -\frac{4\,\pi\,a_s\,\hbar^2}{m^3}\overrightarrow{\nabla}\rho - \overrightarrow{\nabla}\Phi + \frac{\hbar^2}{4\,m^2}\overrightarrow{\nabla}\left[\frac{\nabla^2\rho}{\rho} - \frac{1}{2}\frac{(\nabla\rho)^2}{\rho^2}\right].$$

Thus, the evolution of the shell is governed by three forces,

1. Gravitational attractive force
2. A quantum repulsive force
3. A force arising due to the bosonic interactions which could be attractive ($a_s < 0$) or repulsive ($a_s > 0$).

# Spherical collapse in CDM

# Spherical collapse in CDM

The equation of spherical collapse in CDM can be obtained by taking the limit $\hbar/m \to 0$,

$$\frac{\mathrm{d}^2 \overrightarrow{R}}{\mathrm{d}t^2} = \cancelto{0}{-\frac{4\,\pi\,a_s\,\hbar^2}{m^3}\overrightarrow{\nabla}\rho} - \overrightarrow{\nabla}\Phi + \cancelto{0}{\frac{\hbar^2}{4\,m^2}\overrightarrow{\nabla}\left[\frac{\nabla^2\rho}{\rho} - \frac{1}{2}\frac{(\nabla\rho)^2}{\rho^2}\right]}.$$

For an over dense spherical region containing an arbitrary mass $M$, the above equation becomes,

$$\frac{\mathrm{d}^2 \overrightarrow{R}}{\mathrm{d}t^2} = -\frac{G\,M}{R^2}.$$

We shall assume that the shell is initially expanding along with the Hubble flow. The trajectory of the shell is determined by the first integral of motion, namely,

$$\frac{1}{2}\left(\frac{\mathrm{d}\,R}{\mathrm{d}\,t}\right)^2 - \frac{G\,M}{R} = E.$$

If $E > 0$, the shell will expand for ever with the Hubble flow. On the other hand, if $E < 0$, the shell will eventually stop expanding, turn around and then start collapsing to the center.

# Motion of a spherical shell in CDM

For a shell with $E < 0$, the evolution of radius of the shell $R$ containing mass $M$ is given by[15],

$$\begin{aligned} R &= A\,[\,1 - \cos(\vartheta)\,] \\ t &= B\,[\,\vartheta - \sin(\vartheta)\,] \end{aligned}$$

where $A^3 = G\,M\,B^2$.

Let us now try to understand the behaviour of the solution,

$$\begin{aligned} \text{when}\,\vartheta &= \pi, \quad R(\pi) = R_{max} = 2\,A \\ \text{when}\,\vartheta &= 2\,\pi, \quad R(2\,\pi) = R_{min} = 0. \end{aligned}$$

Thus we see that, a spherical shell containing an overdense region, turns around and collapses to the center, *i.e.* the radius of the shell does not have a lower bound.

[15] T. Padmanabhan, "Structure formation in the universe" (Cambridge University Press, 1993).

# Expression for overdensity in CDM

Let us assume that the background spacetime is EdS. The average density contained in a spherical shell of radius $R$ containing mass $M$ is given by $\bar{\rho} = M/(4\,\pi\,R^3/3)$. In an EdS universe, the background density is given by $\rho_b = 1/(6\,\pi\,\mathrm{G}\;t^2)$.

Hence, the average overdensity inside the spherical shell is

$$1 + \bar{\delta} = \frac{\bar{\rho}}{\rho_b} = \frac{9}{2}\frac{\mathrm{G}\;M\,t^2}{R^3}.$$

Substituting equations for $R$ and $t$, we obtain

$$1 + \bar{\delta} = \frac{9}{2}\frac{(\vartheta - \sin\vartheta)^2}{(1 - \cos\vartheta)^3}.$$

In the linear regime, *i.e.* in the small $\vartheta$ limit,

$$\bar{\delta} \simeq \frac{3\,\vartheta^2}{20} \propto a.$$

# Spherical collapse in FDM

# Power law profile

In order to study the spherical collapse of FDM, we need to assume a density profile for the overdense region.

For simplicity, let us consider a power law density profile of the form,

$$\rho(t,r) = \frac{3-\gamma}{4\pi}\frac{M}{L(t)^3}\left(\frac{r}{L(t)}\right)^{-\gamma},$$

where the normalization factors has been chosen in such a way that, $L(t)$ is the radius of the shell which encloses a mass $M$ and $\gamma$ is a positive number less than $3$ (by demanding that density should be positive).

Assuming that the FDM overdense region maintains such a density profile throughout the evolution, one can derive the equation of motion for the spherical shell.

# Equation of motion for a spherical shell

For a shell of radius $L(t)$, containing mass $M$, the equation of motion can be written as,

$$\frac{\mathrm{d}^2 L}{\mathrm{d}t^2} = \gamma\,(3-\gamma)\frac{a_s\,\hbar^2}{m^3}\frac{M}{L^4} - \frac{\mathrm{G}\;M}{L^2} + \gamma\,(2-\gamma)\frac{\hbar^2}{4\,m^2\,L^3}.$$

As explained before, the evolution of the shell is governed by three forces, namely,

1. the repulsive ($a_s > 0$) or attractive ($a_s < 0$) force due to bosonic self-interaction,
2. attractive gravitational force
3. repulsive quantum force.

Note that, for the power law profile, in order for the quantum force to be positive and non-vanishing, one requires $\gamma < 2$.

# Case of non-interacting Bosons

In the absence of interactions ($a_s = 0$), the equation of motion of the spherical shell can be written as,

$$m \frac{\mathrm{d}^2 L}{\mathrm{d}t^2} = -\frac{k}{L^2} + \frac{l^2}{m\, L^3},$$

where $k = \mathrm{G}\ M\, m$ and $l^2 = (2\,\gamma - \gamma^2)\,\hbar^2/4$.

This equation is mathematically, though not physically, similar to the equation governing the reduced mass in a two-body Kepler problem. Hence, we will draw insights from the solution of Kepler problem to solve the above equation.

Initially, let the overdense shell be expanding along with the Hubble flow. The shell will eventually turn around if the initial value of the first integral of motion of the shell is negative, *i.e.* if,

$$E = \frac{1}{2}\, m \left(\frac{\mathrm{d}L}{\mathrm{d}t}\right)^2 + \frac{l^2}{2\, m\, L^2} - \frac{k}{L} < 0.$$

# Analytical solution

If $E < 0$, the evolution of the shell is given by

$$\begin{aligned} L &= A\,(1 - e\,\cos\vartheta) \\ t &= \left(\frac{m\,A^3}{k}\right)^{1/2} (\vartheta - e\,\sin\vartheta) \end{aligned}$$

where, $A = -k/(2\,E)$ and expression for $e$ is,

$$e = \sqrt{1 + \frac{2\,E\,l^2}{m\,k^2}} = \sqrt{1 + \frac{E\,\hbar^2}{\mathrm{G}^{\,2}\,M^2\,m^3}\,\frac{(2\,\gamma - \gamma^2)}{2}}.$$

Note that, since $E < 0$, the value of $e < 1$. Let us now try to understand the behaviour of the solution,

$$\begin{aligned} \text{when}\,\vartheta &= 0, \quad L(0) = L_{min} = A(1 - e) \\ \text{when}\,\vartheta &= \pi, \quad L(\pi) = L_{max} = A(1 + e). \end{aligned}$$

Since $e < 1$, the radius of the shell is thus bounded from below and hence will oscillate between the two extremum values.

# Numerical evolution

For numerical simulations, it is convenient to rewrite equation of motion in terms of dimensionless variables as,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\tau^2} = \frac{2\,\gamma\, - \,\gamma^2}{4\,y^3} - \frac{1}{y^2}$$

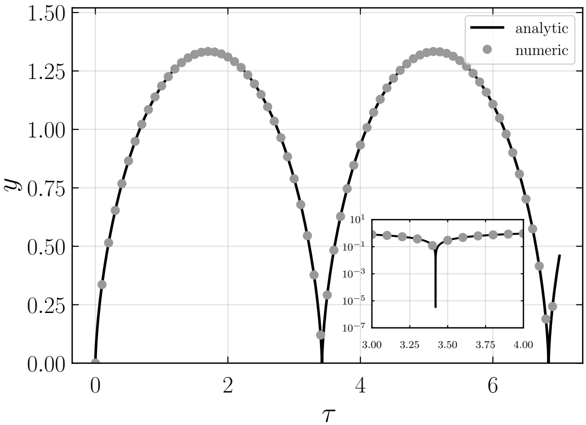
where, we have defined $y = L/L_Q$ and $\tau = t/t_Q$, with $L_Q = \hbar^2/(G\,M\,m^2)$ and $t_Q = \sqrt{L_Q^3/(G\,M)}$.

In order to fix the initial conditions, we assume that,

1. in the beginning, the shell containing an average overdensity of $\bar{\delta}_i = 10^{-5}$, is expanding according to the Hubble flow,
2. the universe is Einstein de Sitter(EdS), *i.e.* the scale factor scales with time as $a \propto t^{2/3}$,
3. we assume that $E < 0$.

For $\gamma = 10^{-10}$, we have numerically solved for $y(\tau)$ and compared with the analytical solutions expressed in terms of $y(\tau)$.

# Numerical evolution




We have assumed that the shell contains a mass $M = 9.1 \times 10^7\,\mathrm{M_{sun}}$, the mass of boson to be $m = 8.1 \times 10^{-23}\,\mathrm{eV}$ and initial values $y_i = 10^{-5}$ and $\bar{\delta}_i = 10^{-5}$. Such a shell would oscillate between $L_{\rm min} = 3.57 \times 10^{-8}\,\mathrm{pc}$ and $L_{\rm max} = 1.9\,\mathrm{kpc}$ .

It is interesting to note that $1 - e = \mathcal{O}(10^{-11})$, hence such a vast difference in $L_{\rm min}$ and $L_{\rm max}$.

## Expression for overdensity

In an EdS spacetime, the average overdensity inside the spherical shell is

$$1 + \bar{\delta} = \frac{\bar{\rho}}{\rho_b} = \frac{9}{2}\frac{\mathrm{G} \;\; M\, t^2}{L^3}.$$

Substituting equations for for $L$ and $t$, we obtain

$$1 + \bar{\delta} = \frac{9}{2}\,\frac{(\vartheta - e\,\sin\vartheta)^2}{(1 - e\,\cos\vartheta)^3}.$$

The above expression for average overdensity within the shell has the following properties:

1. since $e < 1$, the overdensity does not diverge as $\vartheta \to 2\pi$,
2. the averaged overdensity is fluctuating and increasing with time
3. in the limit $\hbar \to 0$, $e \to 1$, it reproduces the CDM expression for averaged overdensity.

# Overdensity in the linear regime

Let us now turn our attention to the behaviour of $\bar{\delta}$ in the small $\vartheta$ limit. Up on Taylor expanding the expression for $\bar{\delta}(\vartheta)$ about $\vartheta \simeq 0$, we obtain

$$1 + \bar{\delta} \simeq \frac{9}{2}\left(\frac{\vartheta^2}{1-e}\right) - \frac{21\,e}{4}\left(\frac{\vartheta^2}{1-e}\right)^2 + \dots .$$

- The above expansion for $\bar{\delta}$ would be valid only if $\vartheta^2 << 1 - e$. However, in this limit, the above expression imply that $\bar{\delta} \simeq -1$ which indicate an underdensity.
- If as we saw in the previous slide, $1 - e$ is very small, then one could first take the limit of $e \to 1$ and then the limit $\vartheta \to 0$. Up on taking the limit in this order we obtain,

$$\bar{\delta} \simeq \frac{3\,\vartheta^2}{20} \propto a,$$

  which is similar to that in CDM.

The above discussion seem to indicate that, in this model, for an overdense region, a sensible small $\vartheta$ limit exists only if the limit $e \to 1$ can be taken before the $\vartheta \to 0$ limit.

# Case of interacting bosons

Due to the lack of analytical solution, we will approach the problem numerically.

For $a_s \neq 0$, we can rewrite equation of motion in dimensionless form as

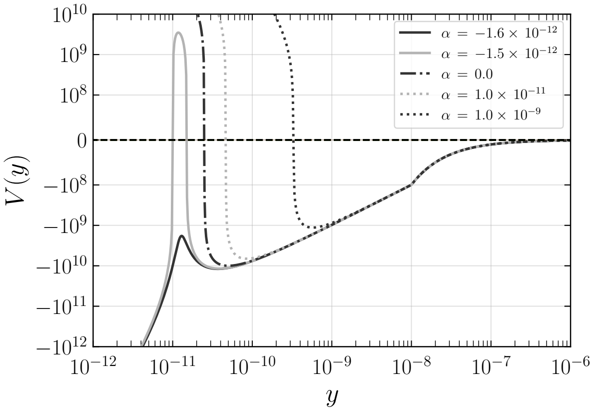
$$\frac{\mathrm{d}^2 y}{\mathrm{d}\tau^2} = \frac{\alpha\,(3\,\gamma - \gamma^2)}{y^4} + \frac{2\,\gamma - \gamma^2}{4\,y^3} - \frac{1}{y^2},$$

where, $a_s \equiv \alpha\,\bar{a}_s$ with $\bar{a}_s = \hbar^2/(G\,M^2\,m)$ and $\alpha$ can be greater than, equal to or less than zero which corresponds to repulsive, nil and attractive interaction respectively.

In order to understand the effect of interactions, it is convenient to look at the form of the effective potential governing the evolution of the shell,

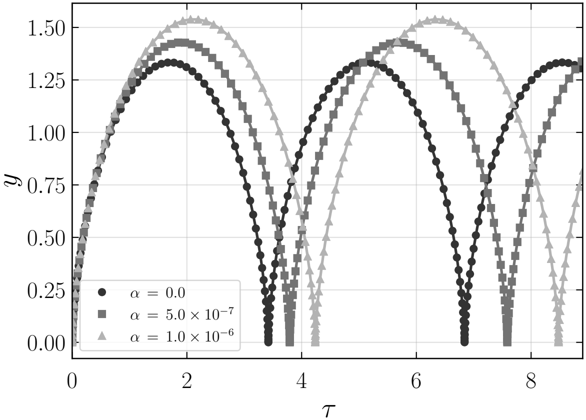
$$V(y) = \frac{\alpha\,(3\,\gamma - \gamma^2)}{3\,y^3} + \frac{2\,\gamma - \gamma^2}{8\,y^3} - \frac{1}{y}.$$

# Effective potential




Horizontal black dashed line denotes the effective energy of the fluid element of the shell with a density profile specified by $\gamma = 10^{-10}$ and with initial conditions $\bar{\delta}_i = 10^{-5}$ and $y_i = 10^{-5}$ and curves denote the effective potential of the fluid element for various values of $\alpha$. As we can see, for $\alpha = -1.6 \times 10^{-12}$, the potential does not have a region which is bounded from both sides and hence the quantum pressure cannot stop the collapse of the shell. For all other values of $\alpha$, shown in the figure, the shell will oscillate.

# Numerical solution




The evolution of the shell for different values of $\alpha > 0$ are shown. The effect of increasing $\alpha$ is a shift in the minimum of the potential to larger values of $y$. This would cause the fluid element to oscillate between larger values of maximum and minimum and with a longer period. The markers indicate the analytical expression.

## Comparison with analytical expression

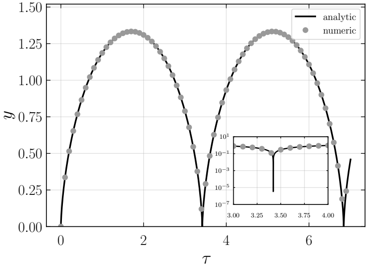
From the previous plot, it appears that the match between the numerical result for $\alpha > 0$ and the analytical result for $\alpha = 0$ is good.

- This is because, in this toy model, the effect of interactions is only felt at small scales where as at large scales, the force is dominated by gravity.
- Though not evident from the plot, the minimum value of $y$ differs from the analytical value as the value of $\alpha$ is non-zero.
- In particular, the analytical expression predicts a minimum radius of $y_{\rm min} = 2.5 \times 10^{-11}$ where as the numerical simulations indicate a minimum radius of $1.5 \times 10^{-11}$, $7.08 \times 10^{-9}$ and $1.0 \times 10^{-8}$ for $\alpha = -1.5 \times 10^{-12}, 5 \times 10^{-7}$ and $10^{-6}$ respectively.
- Hence, one can conclude that, for the parameters that we have considered, the analytical expression derived for non-interacting bosons, holds at large to medium scales for the case of interacting bosons.

# Virialization

# Shell crossing

- The analytical and numerical evaluation of the evolution of a single shell shows that the shell will first expand along with the Hubble flow, then turn around, contract and expand again.
- However, as the shell contracts after turning around, it may interact with other shells, resulting in a complicated dynamics which will not be captured by the simple equations written before.




- But we can assume that the various shells will interact among themselves and will eventually virialize. We can then use virial theorem to gain insights in to the final state of the system.

# Virialization in CDM

# Virial radius

Virial theorem can be used to derive the radius of the spherical overdense region at virialization as follows:

- The total energy of the spherically symmetric overdense region of CDM is given by

$$E_{tot} = T + U_G,$$

where $T$ is the kinetic energy and $U_G$ is the gravitational energy of the system.

- When the system achieves virial equilibrium, the virial theorem states that

$$2T + U_G = 0.$$

which in turn implies that the total energy of the virialized halo is given by $E_{tot} = U_G/2$.

- At turn around, the energy of the system is given by the gravitational energy. Using the fact that energy of the system is conserved and comparing the total energy at turn around and at virialization, one obtains, $L(t_{vir}) = L(t_{ta})/2$, where we have used the expression for gravitational potential energy to be

$$U_G = \frac{3\,\mathrm{G}\ M^2}{5\,L(t)}.$$

# Overdensity at turn around and collapse

Using the expression for radius of the shell at turn around and at virialization we will now compute the overdensity of the system at turn around and virialization in the full and the linear theory.

At turn around, the overdensity in the full theory is given by,

$$1 + \bar{\delta}_{ta} = \frac{9}{2} \frac{(\pi - \sin(\pi))^2}{(1 - \cos(\pi))^3} = \frac{9}{16}\pi^2.$$

At virialization, the overdensity is given by,

$$1 + \bar{\delta}_{vir} = \frac{9\,\mathrm{G}\,M}{2} \frac{t_{vir}^2}{L(t_{vir})^3}.$$

Using the expressions for the radius of the shell at turn around, and hence computing virial radius, $L_{vir}$, using $L_{vir} = L_{ta}/2$, one can compute the overdensity after virialization at $t_{vir} = t(2\,\pi)$ as

$$\begin{aligned}
1 + \bar{\delta}_{vir} &= \frac{9\,\mathrm{G}\,M}{2} \left[ \left( \frac{A^3}{\mathrm{G}\,M} \right)^{1/2} (2\,\pi) \right]^2 \times \frac{1}{A^3} \\
&= 18\,\pi^2.
\end{aligned}$$

# Linear overdensity at turn around

Let us now compute the overdensity in the linear regime. Expanding the expression for $t(\vartheta)$ in the $\vartheta \to 0$ limit, one obtains

$$t \simeq \left(\frac{m\,A^3}{k}\right)^{1/2} \frac{\theta^3}{6}.$$

Using the above expression, one could write an expression for an overdensity at an initial time $t_i$ corresponding to $\vartheta_i$ as

$$\bar{\delta}_i = \frac{3\,\theta_i^2}{20} = \frac{3}{20}\left[6\,\pi\frac{t_i}{t_{ta}}\right]^{2/3}.$$

In an EdS universe, since $\delta \propto a \propto t^{2/3}$ in the linear regime, one could write an expression for $\delta(t)$ as

$$\bar{\delta} \quad \propto \quad \bar{\delta}_i\,\frac{a}{a_i} = \frac{3}{20}\,(6\,\pi)^{2/3}\left(\frac{t}{t_{ta}}\right)^{2/3}.$$

If we use the linear theory to compute the overdensity at turn around, one obtains,

$$\bar{\delta}(t_{ta}) \simeq \frac{3}{20}\,(6\,\pi)^{2/3} = 1.06\,.$$

# Linear overdensity at collapse and critical density

Up on using the linear theory to compute the overdensity after virialization, *i.e.* at $t_{vir} = t(2\pi)$, we get

$$\bar{\delta}(t_{vir}) \simeq \frac{3}{20}\,(12\,\pi)^{2/3} = 1.69\,.$$

The overdensities in the full and the linear theory is summarized below.

| t | Linear theory | Full theory |
|---|---|---|
| turn around | $\bar{\delta}_{ta} = 1.06$ | $\bar{\delta}_{ta} = \frac{9}{16}\,\pi^2 - 1 = 4.55$ |
| virialization | $\bar{\delta}_{vir} \simeq 1.69$ | $\bar{\delta}_{vir} = 18\,\pi^2 - 1 = 176.5$ |

Thus a spherically overdense region of CDM would collapse to form a halo once its linear overdensity becomes the critical value of $\delta_c = 1.69$. Note that, the critical density corresponds to a value of $\delta \simeq 177$ in the full theory.

# Virialization in FDM

## Virial radius

In the case of FDM, virial radius can be derived as follows.

- The total energy of the system is given by,

$$E_{tot} = T + U_Q + U_I + U_G$$

where, $T$ is the kinetic energy of the system, $U_Q$ is the energy stored in the system due to the quantum pressure, $U_I$ is the energy stored in the system due to the interaction and $U_G$ is the gravitational potential energy.

- When the system achieves virial equilibrium, the virial theorem states that [16]

$$2T + 2U_Q + 3U_I + U_G = 0.$$

For non-interacting bosons, the virial theorem hence implies that,

$$T + U_Q = -U_G/2,$$

which in turn implies that the total energy of the virialized halo is given by $E_{tot} = U_G/2$.

- At turn around, the energy of the system is dominated by the gravitational energy. Thus, at turn around, $E_{tot} \simeq U_G$. Using the fact that energy of the system is conserved and comparing the total energy at turn around and at virialization, one obtains, $L(t_{vir}) = L(t_{ta})/2$.

[16] P.-H. Chavanis, Phys. Rev. D **84**, 043531 (2011) .

# Overdensity at turn around and collapse

Using the expression for radius of the shell at turn around and at virialization we will now compute the overdensity of the system at turn around and virialization in the full theory.

At turn around, the overdensity in the full theory is given by,

$$1 + \bar{\delta}_{ta} = \frac{9}{2} \frac{(\pi - e \sin(\pi))^2}{(1 - e \cos(\pi))^3} = \frac{9}{2} \frac{\pi^2}{(1+e)^3}.$$

At virialization, the overdensity is given by,

$$1 + \bar{\delta}_{vir} = \frac{9\,\mathrm{G}\ M}{2} \frac{t_{vir}^2}{L(t_{vir})^3}.$$

Using the expressions for the radius of the shell at turn around, and hence computing virial radius, $L_{vir}$, using $L_{vir} = L_{ta}/2$, one can compute the overdensity after virialization at $t_{vir} = t(2\,\pi)$ as

$$1 + \bar{\delta}_{vir} = \frac{9\,\mathrm{G}\ M}{2} \left[ \left( \frac{A^3}{\mathrm{G}\ M} \right)^{1/2} (2\,\pi) \right]^2 \times \frac{8}{A^3\,(1 + e)^3} = 18\,\pi^2 \frac{8}{(1 + e)^3}.$$

It can be verified that the averaged overdensity in the full theory matches with the CDM value in the $e \to 1$ limit.

# Linear overdensity at turn around

The small $\vartheta$ limit of $\bar{\delta}$ exists only in the limit $e \to 1$. Expanding the expression for $t(\vartheta)$ in the $e \to 1$, $\vartheta \to 0$ limit, one obtains

$$t \simeq \left(\frac{m\,A^3}{k}\right)^{1/2} \frac{e\,\theta^3}{6}.$$

Using the above expression, one could write an expression for an overdensity at an initial time $t_i$ corresponding to $\vartheta_i$ as

$$\bar{\delta}_i = \frac{3\,\theta_i^2}{20} = \frac{3}{20}\left[\frac{6\,\pi}{e}\,\frac{t_i}{t_{ta}}\right]^{2/3}.$$

In an EdS universe, since $\delta \propto a$ in the linear regime, one could write an expression for $\delta(t)$ as

$$\bar{\delta} \quad \propto \quad \bar{\delta}_i\,\frac{a}{a_i} = \frac{3}{20}\left(\frac{6\,\pi}{e}\right)^{2/3}\left(\frac{t}{t_{ta}}\right)^{2/3}.$$

If we use the linear theory to compute the overdensity at turn around, one obtains,

$$\bar{\delta}(t_{ta}) \simeq \frac{3}{20}\left(\frac{6\,\pi}{e}\right)^{2/3} = \frac{1.06}{e^{2/3}}.$$

# Linear overdensity at collapse and critical density

Up on using the linear theory to compute the overdensity after virialization, *i.e.* at $t_{vir} = t(2\,\pi)$, we get

$$\bar{\delta}(t_{vir}) \simeq \frac{3}{20}\left(\frac{12\,\pi}{e}\right)^{2/3} = \frac{1.69}{e^{2/3}}.$$

The overdensities in the full and the linear theory is summarized below.

| t | Linear theory | Full theory |
|---|---|---|
| turn around | $\bar{\delta}_{ta} \simeq \frac{1.06}{e^{2/3}}$ | $\bar{\delta}_{ta} = \frac{9}{2}\,\frac{\pi^2}{(1+e)^3} - 1 \simeq 4.55$ |
| virialization | $\bar{\delta}_{vir} \simeq \frac{1.69}{e^{2/3}}$ | $\bar{\delta}_{vir} = 18\,\pi^2\,\frac{8}{(1+e)^3} - 1 \simeq 176.5$ |

Thus when the averaged linear overdensity inside a spherical shell reaches the critical density, $\bar{\delta}_c \simeq \frac{1.69}{e^{2/3}}$, the overdense region would have collapsed to form a halo.

# Summary and discussion

# Summary

- FDM is a compelling model for dark matter. The quantum nature of FDM which gets manifested at $\mathrm{kpc}$ scales is capable of resolving the small scale issues that has been ailing CDM. FDM halo can be described as a self-gravitating BEC and hence is governed by the GPP equations.

- With the goal of gaining analytical insights in to the evolution of an FDM halo, we investigated the time evolution of a spherical shell containing an overdense region.

- We studied the system in its hydrodynamical form, *i.e.* as a fluid with density $\rho$ and velocity $\overrightarrow{u}$ evolving under the effect of opposing forces of Newtonian gravity and quantum pressure.

- Assuming a spherically symmetric power law profile, we derived the expression for the time evolution of a shell comprising of non-interacting bosons. We verified the analytical results by comparing it with numerics. We found that due to the quantum pressure, the collapse was bounded from below.

## Summary

- Using the analytical expressions, we derived the expression for overdensity of FDM halo in an EdS spacetime and studied its linear regime.
- Further, we numerically studied the evolution of a spherical shell in the presence of interactions and compared the evolution with the case of non-interacting bosons.
- We saw that for repulsive interactions, the effect of stronger interaction is to increase the minima of the potential, which in turn makes the shell oscillate between larger minimum and maximum radius.
- While, in the case of attractive interactions, the shell will oscillate only if the value of $|a_s| << \bar{a}_s$.
- We also found that, for the parameters that we considered, the analytical expression derived for the case of non-interacting bosons is a good approximation at large to medium scales.

## Summary

- As a shell contracts under the effect of gravity after its first turn around, it will interact with inner shells which are expanding again after their initial contraction.

- When they interact, different shells will repel each other due to the quantum pressure and repel or attract each other according to their force of interaction. This would cause the density profile to depart from its initial power law shape. Thus, the shell would now have a more complicated dynamics which is not captured by the equation of motion derived before.

- Nevertheless, since we know that the sphere of FDM would eventually virialize to become a halo, we can use the virial theorem to investigate beyond the validity of the simple equations derived earlier.

- By using virial theorem, we derived the critical density of the overdense region at the time of collapse.

# Discussion

Before we conclude, few remarks are in order.

- First of all, even though this study was motivated by the possibility of FDM being a viable dark matter candidate, most of the analytical calculations discussed in this talk hold for any non-interacting BEC collapsing under the effect of gravity.

- Secondly, in the case of CDM, as the shell is contracting it will cross the shells which is expanding after their first infall. In the case of FDM, however, when two shells come close to each other there will be repulsion due to the quantum pressure and hence the dynamics near shell crossing would be more involved than in CDM.

- Finally, in this work we have used the hydrodynamic description to model the system. It is not clear how well does the hydrodynamic description captures the physics underlying the GPP equations.

It would be interesting to explore these points further.

## Thank you very much for your attention!