Loop contributions to the scalar power spectrum due to quartic order action in ultra slow roll inflation

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Inflation

- Inflation is an epoch of accelerated expansion at the very early stages of the Universe.
- The epoch of inflation solves the shortcomings of the hot Big Bang model, such as the horizon problem and flatness problem.
- It can be achieved by a canonical scalar field ϕ (inflaton) describing by the potential $V(\phi)$:

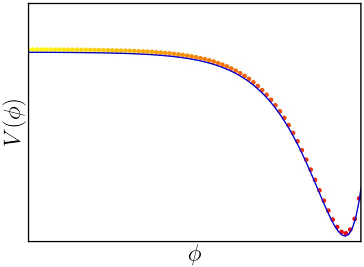
$$\ddot{\phi} + 3H\dot{\phi} + V_{\phi} = 0.$$

- For sufficient inflation → slow roll → field rolling down the potential very slowly.

$$\dot{\phi}^2 \ll V(\phi),$$

$$\ddot{\phi} \ll 3H\dot{\phi},$$

where H is the Hubble parameter→ $H = \dot{a}/a$, and a is the scale factor of the Universe.



- Slow roll parameters:

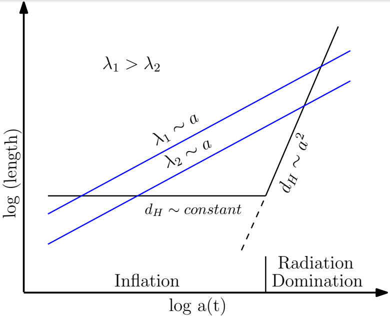
$$\epsilon_1 = -\frac{\dot{H}}{H^2}, \quad \epsilon_{n+1} = \frac{\mathrm{d}\ln|\epsilon_n|}{\mathrm{d}N},$$

where N is e-fold, given by $N = \ln(a)$.

- Slow roll inflation → $\epsilon_1, \epsilon_2, \epsilon_3 ... \ll 1$.

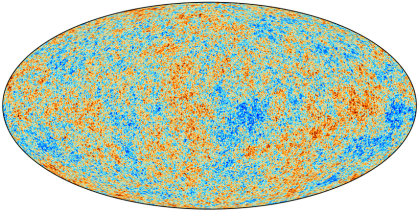
Primordial perturbations

- Other than solving the shortcomings of the hot Big Bang model, inflation provides a natural mechanism to generate perturbations that can satisfy various observations today.
- Inflation ensures that all the physical modes observed today go into the sub-Hubble region in the early time → allowing us to impose the initial condition on the modes.



Different types of perturbations lead to different observables in the present time:

- Scalar perturbations → Anisotropies in CMB, Large scale structures.
- Tensor perturbations → Gravity waves (GWs).


[1]

[1] https://www.esa.int/ESA_Multimedia/Images/2013/03/Planck_CMB

Primordial scalar perturbations

- The primordial perturbations (for scalar, $\zeta(t, \boldsymbol{x})$) are considered to be generated as quantum fluctuations during inflation.
- In Fourier space:

$$\hat{\zeta}_{\boldsymbol{k}}(\eta) = f_k(\eta)\hat{a}_{\boldsymbol{k}}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}} + f_k^*(\eta)\hat{a}_{\boldsymbol{k}}^{\dagger}\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}}.$$

- Equation of motion:

$$f_k''(\eta) + 2\frac{z'}{z}f_k'(\eta) + k^2 f_k(\eta) = 0.$$

- Mukhanov-Sasaki variable for scalar perturbation is defined as $v_k = z\, f_k$, where $z = aM_{_{\mathrm{Pl}}}\sqrt{2\epsilon_1}$. Equation of motion in terms of Mukhanov-Sasaki variable

$$v_k'' + \left(k^2 - \frac{z''}{z}\right)v_k = 0.$$

Power spectrum of the primordial scalar perturbations

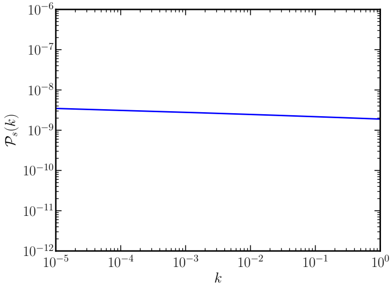
- The power spectrum is defined as two-point correlation of the perturbative modes in Fourier space

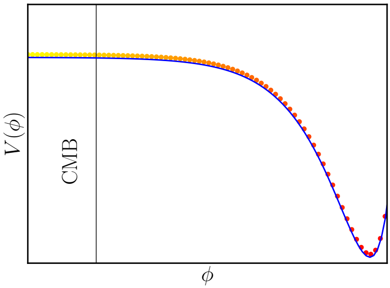
$$\langle f_k f_{k'} \rangle = \frac{2\pi^2}{k^3} \mathcal{P}_{\rm S}(k) \delta^{(3)}(\boldsymbol{k} + \boldsymbol{k}').$$

- In slow roll inflation:

$$\mathcal{P}_{\rm S}(k) = \frac{k^3}{2\pi^2} |f_k|^2 = \frac{H^2}{8\pi^2 M_{\rm Pl}^2 \epsilon_1} \left(\frac{k}{k_*}\right)^{n_s - 1} = A_s \left(\frac{k}{k_*}\right)^{n_s - 1} .$$

- Over large scales → Scalar power spectrum: $A_s \simeq 2.09 \times 10^{-9}, n_s \simeq 0.96$.





Interaction Hamiltonian and in-in formalism

- Expectation value of any operator $\hat{\mathcal{O}}$ can be calculated using in-in formalism

$$\begin{aligned} \left\langle \hat{\mathcal{O}}(\eta) \right\rangle &= \left\langle \mathcal{T} \left(\mathrm{e}^{i \int \mathrm{d}\eta_1 H_{\mathrm{int}}} \, \hat{\mathcal{O}}(\eta) \, \mathrm{e}^{-i \int \mathrm{d}\eta_2 H_{\mathrm{int}}} \right) \right\rangle \\ &= \left\langle \hat{\mathcal{O}}(\eta) \right\rangle_0 - i \int \mathrm{d}\eta' \, \left\langle \mathcal{T} \left[\hat{\mathcal{O}}(\eta), H_{\mathrm{int}}(\eta') \right] \right\rangle \\ &\quad + \int \mathrm{d}\eta_1 \int \mathrm{d}\eta_2 \, \left\langle \mathcal{T} \left[\left[\hat{\mathcal{O}}(\eta), H_{\mathrm{int}}(\eta_1) \right], H_{\mathrm{int}}(\eta_2) \right] \right\rangle + \cdots \end{aligned}$$

  where H_{int} is the Hamiltonian describing the interaction, $\mathcal{T}$ indicates the time ordering.
- In the case of the power spectrum we have calculated $\left\langle \hat{\mathcal{O}}(\eta) \right\rangle_0$ where $\hat{\mathcal{O}} \sim \zeta^2$.
- For three-point correlation $\left\langle \hat{\mathcal{O}}(\eta) \right\rangle_0 \sim \langle \zeta^3 \rangle = 0 \to$ we need to consider the interaction.

- Our interest is to study the loop corrections to the power spectrum $\sim \langle \zeta^2 \rangle$ from the higher order correlations $\sim \zeta^3$, ζ^4...
- There are different methods in the literature to calculate the corrections to the power spectrum.

Method 1: Using the in-in formalism

- Various literature has studied the loop correction to the scalar power spectrum in slow roll inflation using third-order interaction Hamiltonian $H_{\rm int}^{(3)}$[2].
- For correction to the power spectrum the following contribution will be zero:

$$\left\langle \hat{\mathcal{O}}(\eta) \right\rangle_c = -i \int \mathrm{d}\eta' \ \left\langle \mathcal{T}\left[\hat{\mathcal{O}}(\eta), H_{\rm int}^{(3)}(\eta')\right] \right\rangle \sim \langle \hat{\zeta}^5 \rangle = 0.$$

- The dominant contribution to the loop from $H_{\rm int}^{(3)}$ is:

$$\langle \hat{\mathcal{O}}(\eta) \rangle_c = \langle \hat{\mathcal{O}}(\eta) \rangle_{(0,2)} + \langle \hat{\mathcal{O}}(\eta) \rangle_{(2,0)} + \langle \hat{\mathcal{O}}(\eta) \rangle_{(1,1)},$$

  where

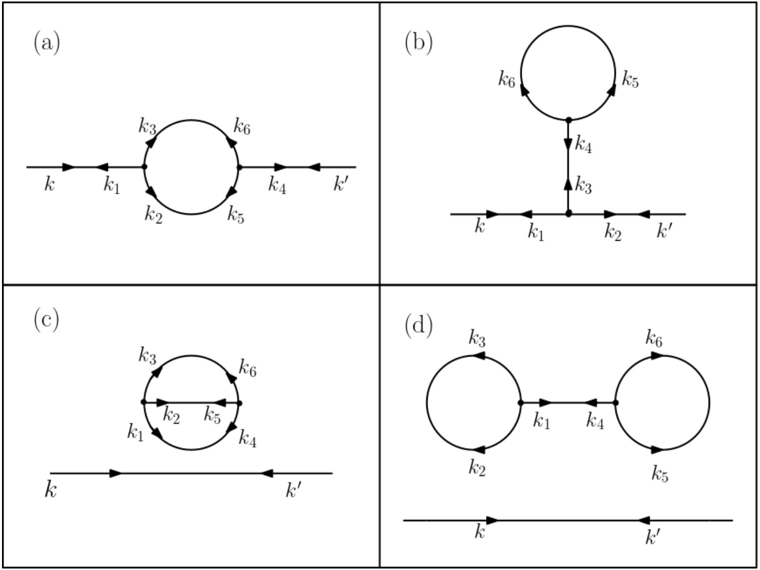
$$\langle \hat{\mathcal{O}}(\eta) \rangle_{(2,0)} = \langle \hat{\mathcal{O}}(\eta) \rangle^{\dagger}_{(0,2)} = -\int_{-\infty}^{\eta} \int_{-\infty}^{\eta} \mathrm{d}\eta_1 \mathrm{d}\eta_2 \langle H_{\rm int}^{(3)}(\eta_1) H_{\rm int}^{(3)}(\eta_2) \hat{\mathcal{O}}(\eta) \rangle,$$

$$\langle \hat{\mathcal{O}}(\eta) \rangle_{(1,1)} = \int_{-\infty}^{\eta} \int_{-\infty}^{\eta} \mathrm{d}\eta_1 \mathrm{d}\eta_2 \langle H_{\rm int}^{(3)}(\eta_1) \hat{\mathcal{O}}(\eta) H_{\rm int}^{(3)}(\eta_2) \rangle.$$

- Here $\langle H_{\rm int}^{(3)\,2} \hat{\mathcal{O}} \rangle \sim \langle \hat{\zeta}^8 \rangle \rightarrow$ one needs to calculate eight-point function.

[2] S. Weinberg, Phys. Rev. D 72 (Aug., 2005),

Method 1: Using the in-in formalism



Method 2: Solving EOM with source term

- One can write the total action governing the perturbations in the following form[3]

$$\begin{aligned} \mathcal{S}[\zeta] &= \mathcal{S}^{(2)}[\zeta] + \mathcal{S}^{(3)}[\zeta] + \cdots \\ &\simeq \int \mathrm{d}^4x \left[z^2 \left\{ \zeta'^2 - \partial\zeta^2 \right\} + g(\eta)\mathcal{G}(\zeta^3) \right] \end{aligned}$$

  where $z = aM_{\rm Pl}\sqrt{2\epsilon_1}$.
- EOM is given by:

$$\tilde{f}_k''(\eta) + 2\frac{z'}{z}\tilde{f}_k'(\eta) + k^2\tilde{f}_k(\eta) = g_1(\eta)\int \frac{\mathrm{d}^3p}{(2\pi)^{3/2}}\,\mathcal{G}(f_p f_{|\boldsymbol{k}-\boldsymbol{p}|})$$

  → the right side of the equation is quadratic in perturbations acting as a source.
- The power spectrum

$$\mathcal{P}_{\rm S}^{0}(k) = \frac{k^3}{2\pi^2}|f_k|^2,$$

$$\mathcal{P}_c(k) = \frac{k^3}{2\pi^2}|\tilde{f}_k|^2.$$

[3] J. Kristiano and J. Yokoyama, arXiv:2211.03395, A. Riotto, arXiv:2301.00599

Method 3: Using the scalar non-Gaussianity parameter $f_{\rm NL}$

- Scalar three-point function is defined as

$$\langle \zeta_{\boldsymbol{k}_1}(\eta_e)\zeta_{\boldsymbol{k}_2}(\eta_e)\zeta_{\boldsymbol{k}_3}(\eta_e)\rangle = \frac{1}{(2\pi)^{3/2}} G_{\zeta\zeta\zeta}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\,\delta^3(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3).$$

- Three-point correlation can be calculated using interaction Hamiltonian:

$$\langle \zeta_{\boldsymbol{k}_1}(\eta_e)\zeta_{\boldsymbol{k}_2}(\eta_e)\zeta_{\boldsymbol{k}_3}(\eta_e)\rangle = i\int \mathrm{d}\eta\,\langle [\zeta_{\boldsymbol{k}_1}(\eta_e)\zeta_{\boldsymbol{k}_2}(\eta_e)\zeta_{\boldsymbol{k}_3}(\eta_e), H^{\rm int}_{\zeta\zeta\zeta}(\eta)]\rangle\,.$$

- The non-Gaussianity parameter is defined as the ratio between the three-point correlation to the two-point correlation

$$f_{\rm NL}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = -\frac{4}{(2\pi^2)^2} G_{\zeta\zeta\zeta}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\left\{\frac{\mathcal{P}_\zeta(k_1)}{k_1^3}\left[\frac{\mathcal{P}_\zeta(k_2)}{k_2^3}+\frac{\mathcal{P}_\zeta(k_3)}{k_3^3}\right]\right\}^{-1}.$$

- Introducing non-linearity to ζ as[4]

$$\zeta(\eta) = \zeta^G(\eta) - \frac{3}{5}\int \frac{\mathrm{d}^3 p}{(2\pi)^{3/2}}\zeta^G_p \zeta^G_{|\boldsymbol{k}-\boldsymbol{p}|} f_{\rm NL}(k,p,|\boldsymbol{k}-\boldsymbol{p}|)$$

[4] C. Unal, Phys. Rev. D 99 (2019), no. 4 041301, P. Adshead, K. D. Lozanov, and Z. J. Weiner, arXiv:2105.01659, H. V. Ragavendra, Phys. Rev. D 105 (2022), no. 6 063533

- The fourth method uses the modification of the Friedmann equations[5].
- The equivalence of the first two methods was studied to some extent, but the implications of all these methods remain to be examined carefully.
- In our work we have focused on the first method, i.e. using the in-in formalism.

- Recent claim by J. Kristiano and J. Yokoyama in arXiv:2211.03395 [hep-th] :
  "...models producing appreciable amount of PBHs generically induce too large one-loop correction on large scale probed by cosmic microwave background radiation. We therefore conclude that PBH formation from single-field inflation is ruled out."
  → Inflationary scenario with a phase of USR [6] on small scales sandwiched between two SR regions.
  → The transitions between the SR-USR are sharp.
  → Leads to a large correction to the power spectrum, which breaks down the perturbation theory.

[5] W.-C. Syu, D.-S. Lee, and K.-W. Ng, Phys. Rev. D 101 (2020), no. 2 025013, S.-L. Cheng, D.-S. Lee, and K.-W. Ng, arXiv:2305.16810, D. Boyanovsky, H. J. de Vega, and N. G. Sanchez, Phys. Rev. D 72 (2005) 103006

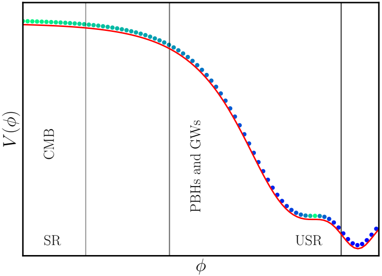
[6] M. J. Morse and W. H. Kinney, [arXiv:1804.01927], M. H. Namjoo, H. Firouzjahi, M. Sasaki, [arXiv:1210.3692 [astro-ph.CO]], J. Martin, H. Motohashi, T. Suyama [arXiv:1211.0083 [astro-ph.CO]]

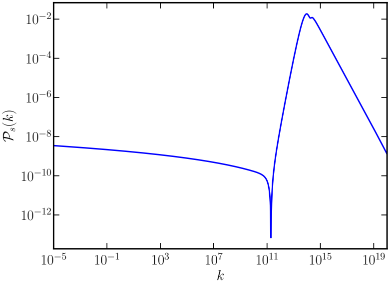
Ultra slow roll inflation

- USR refers to very flat potential, $V_\phi \simeq 0$ leads to EOM

$$\ddot{\phi} + 3H\dot{\phi} \simeq 0.$$

- First slow roll parameter is given by: $\epsilon_1 = \epsilon_{1i}\left[\frac{a_i}{a(\eta)}\right]^6$.
- Second slow roll parameter turns out to be large, and is given by $\epsilon_2 = -6$ (constant), $\rightarrow$ all the higher order slow roll parameters $\epsilon_3, \epsilon_4, \epsilon_5, ... = 0$.





[7].

[7] I. Dalianis, A. Kehagias, and G. Tringas, JCAP01(2019) 037, [arXiv:1805.09483[astro-ph]], I. Dalianis and K. Kritos, [arXiv:2007.07915], H. V. Ragavendra, P. Saha, L. Sriramkumar and J. Silk, [arXiv:2008.12202v2 [astro-ph.CO]]

Solution of scalar modes

- EOM of the Mukhanov-Sasaki variable $v_k = z\, f_k$ is given by

$$v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0,$$

where[8]

$$\frac{z''}{z} = a^2\, H^2 \left(2 - \epsilon_1 + \frac{3\,\epsilon_2}{2} + \frac{\epsilon_2^2}{4} - \frac{\epsilon_1\,\epsilon_2}{2} + \frac{\epsilon_2\,\epsilon_3}{2}\right).$$

- We model these three phases in terms of the evolution of the first slow roll parameter ϵ_1:

$$\epsilon_1(\eta) = \begin{cases} \epsilon_{1\mathrm{i}} & \text{in phase I with } \eta < \eta_1, \\ \epsilon_{1\mathrm{i}}\, (\eta/\eta_1)^6 & \text{in phase II with } \eta_1 < \eta < \eta_2, \\ \epsilon_{1\mathrm{f}} & \text{in phase III with } \eta > \eta_2, \end{cases}$$

- Solution to the Mukhanov-Sasaki variable in the region I with the Bunch-Davies initial condition:

$$v_k^{\mathrm{I}}(\eta) = \frac{1}{\sqrt{2\,k}} \left(1 - \frac{i}{k\,\eta}\right) \mathrm{e}^{-i\,k\,\eta}.$$

[8] J. Martin and L. Sriramkumar, JCAP 01 (2012) 008

Solution of scalar modes

- Solution to the Mukhanov-Sasaki variable in region II and III:

$$v_k^{\rm II}(\eta) = \frac{\gamma_k}{\sqrt{2\,k}}\left(1 - \frac{i}{k\,\eta}\right){\rm e}^{-i\,k\,\eta} + \frac{\delta_k}{\sqrt{2\,k}}\left(1 + \frac{i}{k\,\eta}\right){\rm e}^{i\,k\,\eta}$$

$$v_k^{\rm III}(\eta) = \frac{\alpha_k}{\sqrt{2\,k}}\left(1 - \frac{i}{k\,\eta}\right){\rm e}^{-i\,k\,\eta} + \frac{\beta_k}{\sqrt{2\,k}}\left(1 + \frac{i}{k\,\eta}\right){\rm e}^{i\,k\,\eta}$$

- γ_k, δ_k, α_k, β_k are obtained by matching the solutions at transitions:

$$\gamma_k = 1 + \frac{3\,i}{2\,k\,\eta_1} + \frac{3\,i}{2\,k^3\,\eta_1^3}, \qquad \delta_k = \left(-\frac{3\,i}{2\,k\,\eta_1} - \frac{3}{k^2\,\eta_1^2} + \frac{3\,i}{2\,k^3\,\eta_1^3}\right){\rm e}^{-2\,i\,k\,\eta_1},$$

$$\begin{aligned}
\alpha_k &= \left(1 - \frac{3\,i}{2\,k\,\eta_2} - \frac{3\,i}{2\,k^3\,\eta_2^3}\right)\gamma_k - \left(\frac{3\,i}{2\,k\,\eta_2} - \frac{3}{k^2\,\eta_2^2} - \frac{3\,i}{2\,k^3\,\eta_2^3}\right)\delta_k\,{\rm e}^{2\,i\,k\,\eta_2}, \\
\beta_k &= \left(1 + \frac{3\,i}{2\,k\,\eta_2} + \frac{3\,i}{2\,k^3\,\eta_2^3}\right)\delta_k + \left(\frac{3\,i}{2\,k\,\eta_2} + \frac{3}{k^2\,\eta_2^2} - \frac{3\,i}{2\,k^3\,\eta_2^3}\right)\gamma_k\,{\rm e}^{-2\,i\,k\,\eta_2}.
\end{aligned}$$

Effects due to the transitions

- The second slow roll parameter across the transition

$$\epsilon_2(\eta) = \begin{cases} 0 & \text{in phase I with } \eta < \eta_1, \\ -6 & \text{in phase II with } \eta_1 < \eta < \eta_2, \\ 0 & \text{in phase III with } \eta > \eta_2, \end{cases}$$

- At the first transitions:

$$\epsilon_2' = (\epsilon_2^{\mathrm{II}} - \epsilon_2^{\mathrm{I}})\,\delta^{(1)}(\eta - \eta_1).$$

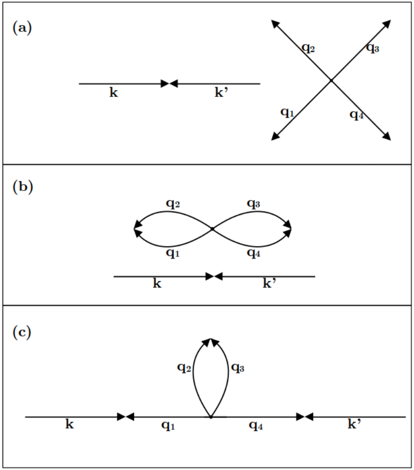
- Calculating the loop correction due to the term

$$H_{\mathrm{int}}^{(3)} = -\frac{M_{\mathrm{Pl}}^2}{2} \int \mathrm{d}^3x \epsilon_1 \epsilon_2' a^2 \,\zeta' \zeta^2$$

  due to the presence of the delta function turns out to be large and the spectral index contains a divergence over the CMB scales
  $\rightarrow$ indicates the breaking of perturbation theory.
- Other claims: The transitions between SR-USR are not sharp $\rightarrow$ corrections are not large enough to break down perturbation theory.
- In all cases the correction $\mathcal{P}_c(k)$ comes out to be positive.

Key points of our work

- We use fourth-order Hamiltonian as it gives leading order corrections ($\sim \langle\zeta^6\rangle$) compared to the correction arrises due to third-order Hamiltonian ($\sim \langle\zeta^8\rangle$).
- We use the term in the Hamiltonian that gives dominant contributions to loop correction due to the transitions between SR-USR.
- We examine the corrections for the onset of USR at different scales.
- We examine the corrections in terms of different parameters
  $\rightarrow$ the duration of the USR,
  $\rightarrow$ the smoothness of the transitions.



Modeling the SR parameters at the transitions

- We write the delta function in Gaussian form

$$\delta^{(1)}(\eta-\tilde{\eta}) \to \frac{1}{\sqrt{\pi}\Delta\eta}\,\mathrm{e}^{-\frac{(\eta-\tilde{\eta})^2}{\Delta\eta^2}},$$

where $\tilde{\eta}$ is the point of transition, $\Delta\eta$ is the smoothness of transition.

- As the second SR parameter is discontinuous across the transitions

$$\epsilon_2' = \begin{cases} \frac{\epsilon_2^{\mathrm{II}}}{\sqrt{\pi}\Delta\eta}\,\mathrm{e}^{-\frac{(\eta-\eta_1)^2}{\Delta\eta^2}} & \text{around } \eta_1, \\ -\frac{\epsilon_2^{\mathrm{II}}}{\sqrt{\pi}\Delta\eta}\,\mathrm{e}^{-\frac{(\eta-\eta_2)^2}{\Delta\eta^2}} & \text{around } \eta_2. \end{cases}$$

- The higher order SR parameters ($\epsilon_{i+1} = aH\epsilon_i'/\epsilon_i$) are:

$$\begin{aligned} \epsilon_3 &= \mp\frac{\eta}{\sqrt{\pi}\Delta\eta}\mathrm{e}^{-\frac{(\eta-\eta_{1,2})^2}{\Delta\eta^2}}, \\ \epsilon_4 &= -\left[1-2\frac{\eta(\eta-\eta_{1,2})}{\Delta\eta^2}\right], \\ \epsilon_5 &= 2\frac{\eta(2\eta-\eta_{1,2})}{\Delta\eta^2-2\eta(\eta-\eta_{1,2})}. \end{aligned}$$

Action describing the fourth-order Hamiltonian

- The dominant term in the quartic action (in spatially flat gauge)[9]:

$$\delta\mathcal{S}_4[\delta\phi] = -\frac{1}{24}\int \mathrm{d}t \int \mathrm{d}^3\boldsymbol{x}\, a^3 V_{\phi\phi\phi\phi}\delta\phi^4\,,$$

  where $V_{\phi\phi\phi\phi} = \mathrm{d}^4V/\mathrm{d}\phi^4$.
- The potential $V = M_{\rm Pl}^2 H^2(3-\epsilon_1)$.
- Writing in terms of SR parameters

$$\begin{aligned}\delta\mathcal{S}_4[\delta\phi] \quad = \quad & \frac{1}{288M_{\rm Pl}^4}\int \mathrm{d}\eta \int \mathrm{d}^3\boldsymbol{x}\, a^4 V\,\frac{\delta\phi^4}{\epsilon_1}\Big[\epsilon_2\epsilon_3\epsilon_4\epsilon_5 + 3\epsilon_2\epsilon_3^2\epsilon_4 + \frac{1}{2}\epsilon_2^2\epsilon_3\epsilon_4 \\ & +3\epsilon_2\epsilon_3\epsilon_4 - 9\epsilon_1\epsilon_2\epsilon_3\epsilon_4 + \epsilon_2\epsilon_3\epsilon_4^2 + \epsilon_2\epsilon_3^3 + \frac{3}{2}\epsilon_2^2\epsilon_3^2 + 3\epsilon_2\epsilon_3^2 - 9\epsilon_1\epsilon_2\epsilon_3^2 \\ & -\frac{1}{2}\epsilon_2^3\epsilon_3 - \frac{3}{2}\epsilon_2^2\epsilon_3 - \frac{35}{2}\epsilon_1\epsilon_2^2\epsilon_3 + 32\epsilon_1^2\epsilon_2\epsilon_3 - 24\epsilon_1\epsilon_2\epsilon_3 - 3\epsilon_1\epsilon_2^3 \\ & +39\epsilon_1^2\epsilon_2^2 - 9\epsilon_1\epsilon_2^2 - 56\epsilon_1^3\epsilon_2 + 72\epsilon_1^2\epsilon_2 + 16\epsilon_1^4 - 48\epsilon_1^3\Big].\end{aligned}$$

- The term $\sim \epsilon_2\epsilon_3\epsilon_4\epsilon_5$ has the largest contribution and it is unique to the term $V_{\phi\phi\phi\phi}$.

[9] E. Dimastrogiovanni and N. Bartolo, JCAP 11 (2008) 016

Gauge choices

- We express the quartic action in terms of ζ (gauge-invariant variable).
- Gauges: $\delta\phi \to$ Spatially flat gauge,
  $\zeta \to$ Uniform density gauge.
- $\delta\phi$ and ζ are related as[10]:

$$\zeta = \frac{H}{\dot{\phi}}\delta\phi = -\frac{\delta\phi}{M_{\rm Pl}\sqrt{2\epsilon_1}}$$

.

- The quartic action in terms of ζ:

$$\mathcal{S}^{(4)}[\zeta] \supset \frac{1}{72}\int \mathrm{d}\eta \int \mathrm{d}^3\boldsymbol{x}\, a^4 V\,(\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5)\,\zeta^4\,.$$

- The corresponding interaction Hamiltonian is given by $H_{\rm int}^{(4)} = \pi_\zeta\dot{\zeta} - \mathcal{L}^{(4)}$, where $\pi_\zeta \to$ conjugate momentum of ζ:

$$H_{\rm int}^{(4)} \supset -\frac{1}{72}\int \mathrm{d}^3\boldsymbol{x}\, a^4 V(\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5)\zeta^4(\eta,\boldsymbol{x})\,.$$

[10] E. Dimastrogiovanni and N. Bartolo, JCAP 11 (2008) 016

The correction to power spectrum due to the term considered

- The correction due to the term of interest in $H_{\rm int}^{(4)}$ at the leading order:

$$\begin{aligned}\langle 0|\hat{\zeta}_{\boldsymbol{k}}(\eta_e)\hat{\zeta}_{\boldsymbol{k}'}(\eta_e)|0\rangle \quad &\simeq \quad \langle 0|\hat{\zeta}_{\boldsymbol{k}}(\eta_e)\hat{\zeta}_{\boldsymbol{k}'}(\eta_e)|0\rangle \\ &\quad -i\langle 0|\left[\hat{\zeta}_{\boldsymbol{k}}(\eta_e)\hat{\zeta}_{\boldsymbol{k}'}(\eta_e)\,,\int \mathrm{d}\eta\,\mathcal{T}\left(\hat{H}_{\rm int}^{(4)}(\eta,\boldsymbol{x})\right)\right]|0\rangle.\end{aligned}$$

- The correction to the two-point function has the form

$$\begin{aligned}\langle 0|\hat{\zeta}_{\boldsymbol{k}}\hat{\zeta}_{\boldsymbol{k}'}(\eta_e)|0\rangle_{\rm C} \quad &\simeq \quad \frac{i}{6}\,f_k(\eta_e)f_{k'}(\eta_e)\delta^{(3)}(\boldsymbol{k}+\boldsymbol{k}')\int \mathrm{d}\eta\, a^4 V\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5 f_k^*(\eta)f_{k'}^*(\eta) \\ &\quad \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}|f_q(\eta)|^2 + \text{ complex conjugate}\,.\end{aligned}$$

- Dimensionless power spectrum, which we shall denote as $\mathcal{P}_{\rm C}^{(4)}(k)$, has the form

$$\begin{aligned}\mathcal{P}_{\rm C}^{(4)}(k) \quad &= \quad \frac{i}{6}\left(\frac{k^3}{2\pi^2}\right)f_k^2(\eta_e)\int \mathrm{d}\eta\, a^4 V\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5\,[f_k^*(\eta)]^2\int \mathrm{d}\ln q\,\mathcal{P}_{\rm S}(q,\eta) \\ &\quad + \text{complex conjugate}.\end{aligned}$$

In the case of slow roll

- The mode function $f_k(\eta)$ can be expressed as:

$$f_k(\eta) = -\frac{H\eta}{M_{\rm Pl}\sqrt{4k\epsilon_1}}\mathrm{e}^{-ik\eta}\left(1-\frac{i}{k\eta}\right).$$

- The corrected power spectrum can be written as:

$$\mathcal{P}_{\rm C}^{(4)}(k) = -\frac{i}{8}\epsilon_2\epsilon_3\epsilon_4\epsilon_5\left(\mathcal{P}_{\rm S}^{0}\right)^2\int\frac{\mathrm{d}\eta}{k\,\eta^2}\left(1+\frac{i}{k\eta}\right)^2\mathrm{e}^{2ik\eta}\int\mathrm{d}\ln q\,(1+q^2\eta^2)$$
$$+\ \text{complex conjugate},$$

where $\mathcal{P}_{\rm S}^{0} = H^2/(8\pi^2 M_{\rm Pl}^2\epsilon_1) \simeq 2.1\times 10^{-9}$.

- Approximations: $3H^2M_{\rm Pl}^2 \simeq V$ and $a=-1/(H\eta)$.
$\epsilon_1 \simeq \epsilon_2 \simeq \epsilon_3 \simeq \epsilon_4 \simeq \epsilon_5 \simeq 10^{-3}$.

In the case of slow roll

- After momentum integration:

$$\mathcal{P}_{\mathrm{C}}^{(4)}(k) = -\frac{i}{8}\epsilon_1^4\left(\mathcal{P}_{\mathrm{S}}^{0}\right)^2\int_{x_{\mathrm{i}}}^{x_{\mathrm{e}}}\frac{\mathrm{d}x}{x^2}\left(1+\frac{i}{x}\right)^2\mathrm{e}^{2ix}\left[c_1+\frac{c_2}{2}\,x^2\right] + \text{complex conjugate}\,.$$

where $x_j = k\eta_j$ and

$$c_1 = \ln\left(\frac{k_{\mathrm{max}}}{k_{\mathrm{min}}}\right) = \ln\left(\frac{\eta_i}{\eta_e}\right) \simeq 70\,,$$

$$c_2 = \frac{(k_{\mathrm{max}}^2 - k_{\mathrm{min}}^2)}{k^2} \simeq \left(\frac{k_{\mathrm{max}}}{k}\right)^2 \simeq 10^{42}.$$

- After integrating over time:

$$\mathcal{P}_{\mathrm{C}}^{(4)}(k) \simeq \frac{1}{8}\epsilon_1^4\left(\mathcal{P}_{\mathrm{S}}^{0}\right)^2\left[\left(\frac{28}{9}-\frac{4\gamma}{3}-\frac{4}{3}\ln(-2x_{\mathrm{e}})\right)c_1 + 3\,c_2\right].$$

- Ignoring $c_2 \to$ contribution due to sub-Hubble modes:

$$\mathcal{P}_{\mathrm{C}}^{(4)}(k) \simeq -\frac{1}{6}\epsilon_1^4\left(\mathcal{P}_{\mathrm{S}}^{0}\right)^2\,c_1\,\ln\left(2\frac{k_*}{k_{\mathrm{max}}}\right) \sim 10^{-28}\,.$$

Model of interest : SR-USR-SR

- Using the form of the SR parameters and after time integration we get the contribution comes at the two transitions:

$$\mathcal{P}_{\mathrm{C}}^{(4)}(k) = i\,\frac{M_{\mathrm{Pl}}^2}{H^2}\,\frac{\epsilon_{1_{\mathrm{i}}}\epsilon_2^{\mathrm{II}}}{\Delta\eta^2}\,\frac{k^3}{2\pi^2}\,f_k^2(\eta_e)\left[\frac{[\zeta_k^*(\eta_1)]^2}{\eta_1}\int_{k_{\min}}^{k_1}\mathrm{d}\ln q\,\mathcal{P}_{\mathrm{S}}(q,\eta_1) - \left(\frac{\eta_2}{\eta_1}\right)^6\frac{[f_k^*(\eta_2)]^2}{\eta_2}\int_{k_{\min}}^{k_2}\mathrm{d}\ln q\,\mathcal{P}_{\mathrm{S}}(q,\eta_2)\right] + \text{complex conjugate}\,.$$

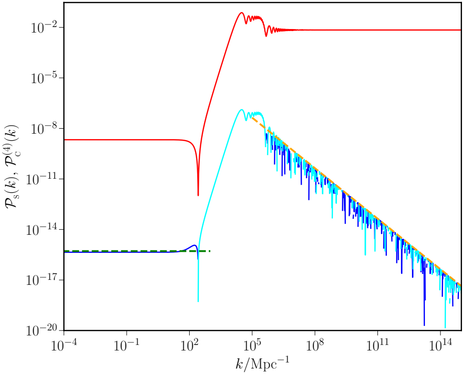
- After momentum integration on the super-Hubble modes:

$$\mathcal{P}_{\mathrm{C}}^{(4)}(k) \simeq \frac{i}{4}\left(\frac{H^2}{8\pi^2 M_{\mathrm{Pl}}^2\epsilon_{1_{\mathrm{f}}}}\right)^2\frac{\epsilon_2^{\mathrm{II}}}{k^3\eta_2\Delta\eta^2}\,\mathcal{F}^2(\alpha_k,\beta_k,\eta_e) \times\Bigg\{\left(\mathcal{F}^*(1,0,\eta_1)\right)^2\left(\frac{k_1}{k_2}\right)^7\left[\ln\left(\frac{k_1}{k_{\min}}\right)+\frac{1}{2}\left(1-\frac{k_{\min}^2}{k_1^2}\right)\right] - \left(\mathcal{F}^*(\alpha_k,\beta_k,\eta_2)\right)^2\Bigg[\left(\frac{k_1}{k_2}\right)^6\left[\ln\left(\frac{k_2}{k_{\min}}\right)-\frac{1}{10}\left(1-\frac{k_{\min}^2}{k_2^2}\right)\right] - \left[\frac{2}{5}\left(\frac{k_1}{k_2}\right)-\left(\frac{k_1}{k_2}\right)^4\right]\left[1-\left(\frac{k_{\min}}{k_2}\right)^2\right]\Bigg]\Bigg\} + \text{complex conjugate}\,,$$

where

$$\mathcal{F}(\alpha_k,\beta_k,\eta) = \alpha_k\,\mathrm{e}^{-ik\eta}(k\eta-i)+\beta_k\,\mathrm{e}^{ik\eta}(k\eta+i)\,.$$

Corrected power spectrum



The first order power spectrum $\mathcal{P}_{\rm S}(k)$ (in red) and the loop correction $\mathcal{P}_{\rm C}^{(4)}(k)$ (in blue) are displayed for the following parameters: $H = 1.3 \times 10^{-5}\, M_{\rm Pl}, \epsilon_{1_{\rm i}} = 10^{-3}$, $\epsilon_2^{\rm II} = -6$, $\Delta\eta = 10^{-3}\,\mathrm{Mpc}$, $k_1 = 10^4\,\mathrm{Mpc}^{-1}$, $\Delta N = 2.5$, $k_{\rm min} = 10^{-6}\,\mathrm{Mpc}^{-1}$, $k_{\rm max} = 10^{20}\,\mathrm{Mpc}^{-1}$. The negative values of $\mathcal{P}_{\rm C}^{(4)}(k)$ are shown in cyan.

Asymptotic behaviours of $\mathcal{P}_{\mathrm{C}}^{(4)}(k)$

- Correction in the limit $k \ll k_1 < k_2$:

$$\frac{\mathcal{P}_{\mathrm{C}}^{(4)}(k \ll k_1)}{\mathcal{P}_{\mathrm{S}}(k \ll k_1)} = -\frac{2\,\epsilon_2^{\mathrm{II}}}{3\,k_1^2 \Delta\eta^2} \left(\frac{H^2}{8\pi^2 M_{\mathrm{Pl}}^2 \epsilon_{1_{\mathrm{i}}}}\right) \left(\frac{k_2}{k_1}\right)^3 \ln\left(\frac{k_1}{k_{\mathrm{min}}}\right).$$

- Correction in the limit $k \gg k_2 > k_1$:

$$\begin{aligned}\mathcal{P}_{\mathrm{C}}^{(4)}(k \gg k_2) \quad \simeq \quad & -\frac{\epsilon_2^{\mathrm{II}}}{2kk_2\Delta\eta^2} \left(\frac{H^2}{8\pi^2 M_{\mathrm{Pl}}^2 \epsilon_{1_{\mathrm{f}}}}\right)^2 \left\{ \left[\ln\left(\frac{2k_2}{k_1}\right) + \gamma - \frac{1}{6}\right] \sin\left(\frac{2k}{k_2}\right) \right. \\ & \left. - \left(\frac{k_1}{k_2}\right)^5 \left[\ln\left(\frac{k_1}{k_{\mathrm{min}}}\right) + \frac{1}{2}\right] \sin\left(\frac{2k}{k_1}\right) \right\},\end{aligned}$$

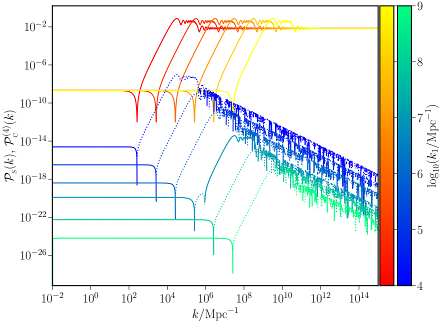
  where $\gamma \simeq 0.577$ is the Euler-Mascheroni constant.

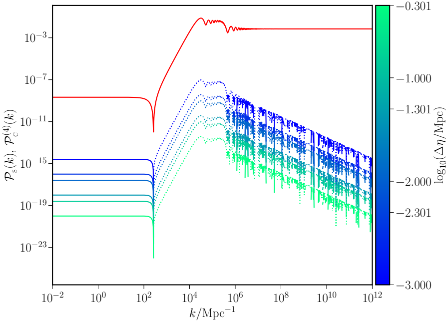
- The ratio of $\mathcal{P}_{\mathrm{C}}^{(4)}(k)$ to $\mathcal{P}_{\mathrm{S}}(k)$ in this regime is

$$\frac{\mathcal{P}_{\mathrm{C}}^{(4)}(k \gg k_2)}{\mathcal{P}_{\mathrm{S}}(k \gg k_2)} \simeq -\frac{\epsilon_2^{\mathrm{II}}}{2\,k\,k_2\,\Delta\eta^2} \left(\frac{H^2}{8\pi^2 M_{\mathrm{Pl}}^2 \epsilon_{1_{\mathrm{f}}}}\right) \left[\ln\left(\frac{2\,k_2}{k_1}\right) + \gamma - \frac{1}{6}\right]$$

- We used $\mathcal{P}_{\mathrm{S}}(k \ll k_1) = H^2/(8\pi^2 M_{\mathrm{Pl}}^2 \epsilon_{1_{\mathrm{i}}})$ and $\mathcal{P}_{\mathrm{S}}(k \gg k_2) = H^2/(8\pi^2 M_{\mathrm{Pl}}^2 \epsilon_{1_{\mathrm{f}}})$.

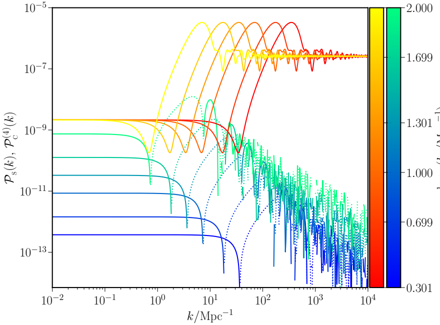
Corrections for late onset of USR

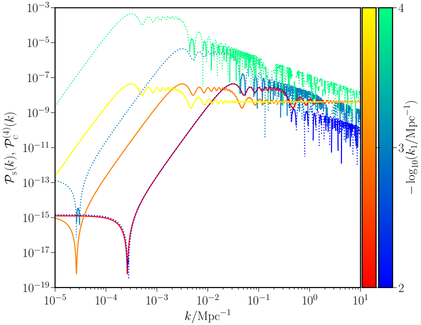




- Required for enhancement of power over small scales → production of PBHs.
- Onset of USR → $k_1 = 10^4,\ 10^5,\ 10^6,\ 10^7,\ 10^8$ and $10^9\ \mathrm{Mpc}^{-1}$.
- Duration $\Delta N = 2.5$, Smoothness $\Delta\eta = 10^{-3}\ \mathrm{Mpc}$.
- Smoothness values: $\Delta\eta = 0.5,\ 0.1,\ 5\times10^{-2},\ 10^{-2},\ 5\times10^{-3},$ and $10^{-3}\ \mathrm{Mpc}$.
- Increasing smoothness → decreasing $\mathcal{P}_C^{(4)}(k)$.
- Corrections are small → PBH production in single field inflation is not ruled out.

Corrections for onset of USR at other scales

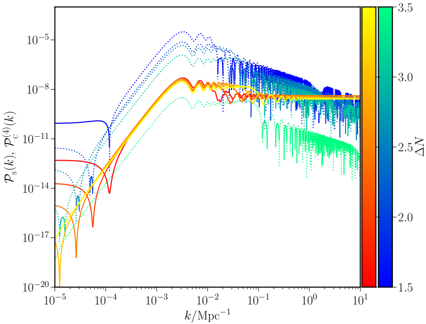


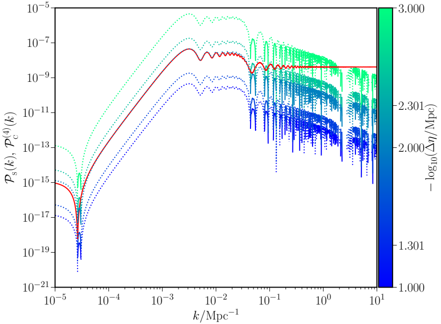


- Intermediate onset of USR → USR phase just after the CMB scales ($k1 > 0.2\,{\rm Mpc}^{-1}$).
- Duration is fixed such that $\mathcal{P}_{\rm S}(k) < 10^{-5}$ → imposed by FIRAS due to CMB spectral distortions.
- For $k_1 = 2\,{\rm Mpc}^{-1}$, the correction is nearly 30% of $\mathcal{P}_{\rm S}(k)$ over the CMB scales.

- Early onset of USR ($k_1 \sim 10^{-2}\,{\rm Mpc}^{-1}$) → suppressing scalar power over large scales.
- $\mathcal{P}_{\rm C}^{(4)}(k)$ dominates over $\mathcal{P}_{\rm S}(k)$ → breakdown of perturbation theory.
- Total power spectrum $\mathcal{P}_{\rm S}(k) + \mathcal{P}_{\rm C}^{(4)}(k)$ becomes negative.

Corrections for early onset of USR: effects of duration and smoothness





- $k_1 = 10^{-3}\,\mathrm{Mpc}^{-1}$ and $\Delta\eta = 10^{-3}\,\mathrm{Mpc}$.
- $\Delta N = 1.5,\ 2,\ 2.5,\ 3,\ 3.5$ e-folds.
- H and ϵ_1 adjusted to have $\mathcal{P}_{\rm S}(k) \sim 10^{-9}$ at pivot scale.
- $\mathcal{P}_{\rm C}^{(4)}(k)$ decreases as ΔN increases.

- $k_1 = 10^{-3}\,\mathrm{Mpc}^{-1}$ and $\Delta N = 2.5$.
- $\Delta\eta = 0.5,\ 0.1,\ 5\times10^{-2},\ 10^{-2},\ 5\times 10^{-3},\ 10^{-3}\,\mathrm{Mpc}$
- $\mathcal{P}_{\rm C}^{(4)}(k)$ increases as $\Delta\eta$ increases.

On the divergence in sub-Hubble regime

- In real space:

$$\langle\hat{\zeta}(\eta,\boldsymbol{x})\hat{\zeta}(\eta,\boldsymbol{x}')\rangle \simeq \int_{-\infty}^{k_\Delta} \mathrm{d}\ln q\, \mathcal{P}_{\mathrm{S}}(q,\eta)\,.$$

- In SR inflation focussing on the term leading to the quadratic divergence, we have:

$$\langle\zeta(\eta,\boldsymbol{x})\zeta(\eta,\boldsymbol{x}')\rangle \simeq \frac{H^2}{16\pi^2 M_{\mathrm{Pl}}^2\epsilon_1} k_\Delta^2\eta^2 \,, \simeq \frac{L_{\mathrm{Pl}}^2}{(2\pi a\,|\boldsymbol{x}-\boldsymbol{x}'|)^2}\frac{1}{4\epsilon_1}\,,$$

  where $L_{\mathrm{Pl}} \equiv 1/M_{\mathrm{Pl}}$.
- Two-point correlation of a free, massless scalar field in FLRW spacetime:

$$\langle\delta\phi(\eta,\boldsymbol{x})\delta\phi(\eta,\boldsymbol{x}')\rangle \simeq \frac{1}{(2\pi\, a|\boldsymbol{x}-\boldsymbol{x}'|)^2}\,.$$

- Divergence occurs for:
  1. $|\boldsymbol{x}-\boldsymbol{x}'| \to 0$
  2. $a \to 0$
- The divergences of such a system can be removed by appropriate regularization in the loop expansion.

Structure of $\mathcal{P}_{\rm C}^{(4)}(k)$

- The correction to two-point correlation:

$$\langle\zeta(\eta_e,\boldsymbol{x}_1)\zeta(\eta_e,\boldsymbol{x}_2)\rangle_{\rm C} = -12\,i\int {\rm d}\eta\,\lambda(\eta)\int {\rm d}^3\boldsymbol{k}_1\int {\rm d}^3\boldsymbol{k}_2\,\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2)\, f_{k_1}(\eta_e)f_{k_2}(\eta_e)f_{k_1}^*(\eta)f_{k_2}^*(\eta) \times {\rm e}^{i(\boldsymbol{k}_1\cdot\boldsymbol{x}_1+\boldsymbol{k}_2\cdot\boldsymbol{x}_2)}\int\frac{{\rm d}^3\boldsymbol{q}}{(2\pi)^6}\,|f_q(\eta)|^2 + \text{complex conjugate}\,.$$

- We decompose the mode functions into amplitudes and phases:

$$f_k(\eta) = |f_k(\eta)|{\rm e}^{i\,\theta}\,,$$

where the $|f_k(\eta)|$ is the positive definite amplitude and

$$\theta(k,\eta) = \tan^{-1}\left[\frac{\mathfrak{Im}\,[f_k(\eta)]}{\mathfrak{Re}\,[f_k(\eta)]}\right] = \tan^{-1}\left[\frac{\sin(k\eta)+\cos(k\eta)/(k\eta)}{\sin(k\eta)/(k\eta)-\cos(k\eta)}\right].$$

- Correction is given by:

$$\mathcal{P}_{\rm C}^{(4)}(k) = 24\,\mathcal{P}_{\rm S}(k,\eta_e)\int {\rm d}\eta\,\lambda(\eta)|f_k(\eta)|^2\int {\rm d}\ln q\,\mathcal{P}_{\rm S}(q,\eta)\,\sin\left[2\left[\theta(k\eta_e)-\theta(k\eta)\right]\right].$$

The phase factor explains the negative sign of $\mathcal{P}_{\rm C}^{(4)}(k)$.

Summary

1. We calculated the loop contribution to the scalar power spectrum due to the quartic order action in models allowing a brief epoch of USR.

2. We have calculated the correction for different onsets of USR:
   Late onset $\rightarrow$ the correction is small.
   Intermediate onset $\rightarrow$ the correction can be 30% of $\mathcal{P}_{\mathrm{S}}(k)$.
   Early onset $\rightarrow$ the correction is comparable or can dominate over $\mathcal{P}_{\mathrm{S}}(k)$.

3. We have investigated the correction also as function of the duration of USR, ΔN and the smoothness of the transitions, $\Delta\eta$:
   $\rightarrow$ For early onset of USR $\mathcal{P}_{\mathrm{C}}^{(4)}(k)$ decreases with increasing ΔN.
   $\rightarrow$ $\mathcal{P}_{\mathrm{C}}^{(4)}(k)$ decreases with increasing $\Delta\eta$.

4. The USR models with a late onset of USR that can produce PBHs are not ruled out, rather an early onset of USR leads to the breakdown of perturbation theory.

5. Unlike $\mathcal{P}_{\mathrm{S}}(k)$, $\mathcal{P}_{\mathrm{C}}^{(4)}(k)$ can be negative.

6. The divergences due to the contributions from the sub-Hubble regime need to be removed with appropriate regularization and renormalization.

THANK YOU