Constraining the cosmic curvature density parameter

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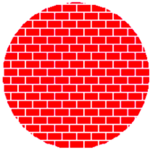
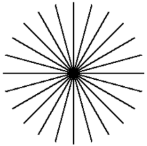
Cosmological Principle

- The universe is **spatially homogeneous** and **isotropic** on large scales.

- **Homogeneity** - The universe looks the same at each point.

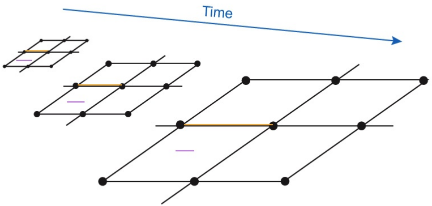




- **Isotropy** - The universe looks similar in all directions.

Comoving Coordinates

- Physical distance, $r(t) = a(t)x$

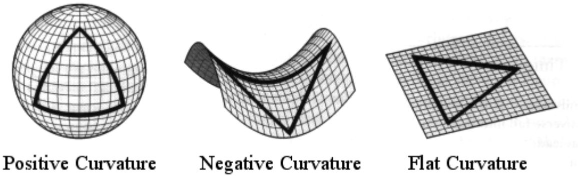


- Comoving distance, x.
- Scale factor, $a(t) \rightarrow$ measure of expansion.
- For convenience: At present time $t_0 \rightarrow \boxed{a(t_0) = a_0 = 1}$.

Friedmann-Lemaître-Robertson-Walker metric

$$ds^2 = -\,dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2\right].$$

- **scale factor**, $a(t)$: determined by the field equations.
- **curvature index**, $k \to \{+1,\ 0, -1\}$: gives the **spatial geometry**.



$k = +1$	$k = -1$	$k = 0$
closed 3 sphere	open 3 hyperboloid	infinite plane

Standard Cosmological Model

- **Cosmological Principle**

- **General Relativity:**

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}.$$

- **Hydrodynamic approximation:**

$$\begin{aligned} T_{\mu\nu} &= (\rho + p) u_\mu u_\nu + p g_{\mu\nu}, \\ w &= p/\rho. \end{aligned}$$

Friedmann Cosmology

Friedmann Equation:

$$3H^2 + 3\frac{k}{a^2} = 8\pi G\rho. \tag{1}$$

Hubble parameter:

$$H = \frac{\dot{a}}{a}.$$

Density parameter:

$$\Omega = \frac{\rho}{\rho_c}; \quad \rho_c = \frac{3H^2}{8\pi G} \rightarrow \textbf{critical density}.$$

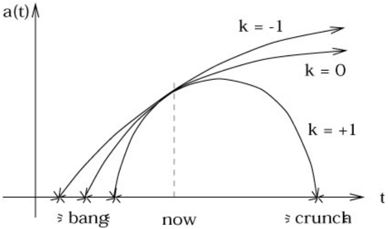
- Eq (1) takes the form:

$$\Omega - 1 = \frac{k}{a^2H^2}.$$

- **Curvature density parameter**:

$$\Omega_k = -\frac{k}{a^2H^2}, \quad \text{such that } \Omega + \Omega_k = 1.$$

Curvature density parameter



$$\Omega_k = -\frac{k}{a^2 H^2}; \ \Omega + \Omega_k = 1.$$

It indicates the fate of the universe.

- flat: $k = 0 \rightarrow \Omega_k = 0; \ \Omega = 1 \implies$ critical expansion,
- open: $k = -1 \rightarrow \Omega_k > 0; \ \Omega < 1 \implies$ continued forever expansion,
- closed: $k = +1 \rightarrow \Omega_k < 0; \ \Omega > 1 \implies$ Big Crunch.

Motivation

Fine-tuning problem: Initial value of Ω_k is very close to 0 $\rightarrow$ taken care of by **inflation**.

From Friedmann Eq.,

$$\Omega - 1 = \frac{k}{a^2 H^2}.$$

Stability analysis:

$$\dot{\Omega} = \tfrac{2k}{a^2 H} q,$$

$$\implies \quad \boxed{\dot{\Omega} = 2qH(\Omega - 1)}.$$

Here, $q = -\frac{\ddot{a}}{aH^2}$ $\rightarrow$ **deceleration parameter**.

For $q > 0$: If $\Omega > 1 \implies \dot{\Omega} > 0$;
If $\Omega < 1 \implies \dot{\Omega} < 0$.

For $q < 0$: If $\Omega > 1 \implies \dot{\Omega} < 0$;
If $\Omega < 1 \implies \dot{\Omega} > 0$.

$\therefore \Omega = 1 \rightarrow$ **stable** if $q < 0$ (accelerated expansion).

If $\Omega = 1 \implies$ it remains so for all time.

But what if it is not?

matter-dominated universe

$$
\begin{aligned}
a^2H^2 &\propto t^{-2/3}; \\
|\Omega - 1| &\propto t^{2/3}.
\end{aligned}
$$

radiation-dominated universe

$$
\begin{aligned}
a^2H^2 &\propto t^{-1}; \\
|\Omega - 1| &\propto t.
\end{aligned}
$$

$|\Omega - 1|$: an increasing function of time.

If $\Omega \to 1$ today, it had to be much closer to 1 in the past!

Any deviation from flatness: the universe will become more curved.

Therefore, if Ω_k is negligible but k itself is non-zero, it may reappear in course of evolution and make its presence felt as the universe evolves!

Reconstruction of some dark energy parameters indicate that a non-flat space section may not be easily ruled out...

Few Examples:

- Using Ω_{k0} as free parameter affects reconstruction of DE EoS w [C. Clarkson, M. Cortes & B. A. Bassett, JCAP **08** (2007) 011]

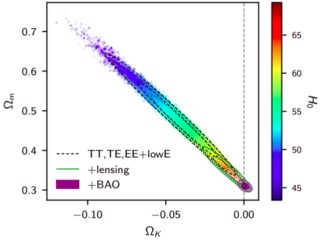
- Reconstruction of the deceleration parameter q [Y.-G. Gong & A. Wang, Phys. Rev. D **75**, 043520 (2007)]:
  - $-0.064 \leq \Omega_{k0} \leq 0.028$: ΛCDM model.
  - $|\Omega_{k0}| < 0.05$: one-parameter DE model $\rightarrow w(z) = \frac{w_0}{1+z} \exp\left(\frac{z}{1+z}\right)$.

    [Y. G. Gong & Y. Z. Zhang, Phys. Rev. D **72**, 043518 (2005).]

  where, $z \rightarrow$ cosmological redshift: $\boxed{1 + z = \frac{a_0}{a};\ a_0 = 1.}$

Planck 2018 Constraints

Ω_{k0}	Datasets
$-0.056^{+0.028}_{-0.018}$	Planck TT+lowE
$-0.044^{+0.018}_{-0.015}$	Planck TT,TE,EE+lowE
-0.0106 ± 0.0065	TT,TE,EE+lowE+lensing
0.0007 ± 0.0019	TT,TE,EE+lowE+lensing+BAO



[N. Aghanim et al. [Planck Collaboration], A&A **641**, A6 (2020)]

Reconstruction

Parametric approach:

- Assume a background model: $H \equiv H(z, \Omega_{k0}, \{\theta\})$.
- Constrain Ω_{k0} with the model parameters $\{\theta\}$.

Null-test approach:

$$\Omega_k = \frac{[E(z)D'(z)]^2 - 1}{D^2(z)}$$

[C. Clarkson et al. JCAP **08** (2007) 011]

$$\mathcal{O}_k = E(z)D'(z) - 1.$$

$E(z) = H(z)/H_0 \rightarrow$ reduced Hubble parameter, $D(z) \rightarrow$ normalized transverse comoving distance.

Non-parametric approach:

- No parametric model for the expansion history assumed.
- No dependence on the theory of gravity or matter distribution.
- Basic assumption: FLRW metric.
- Constraints on Ω_{k0}: obtained directly from data.

Gaussian Process

- Building blocks for Gaussian Processes: **Gaussian distribution**.
- A **multivariate** Gaussian distribution $\rightarrow$ **joint distribution is Gaussian**.
- GP characterised by: **mean** function, $\mu(z)$ and **covariance** function, $\kappa(z, \tilde{z})$.

$$f(z) \sim \mathrm{GP}\left(\mu(z), \kappa(z, \tilde{z})\right) . \tag{2}$$

$$\begin{aligned} \text{Mean function:} \qquad \mu(z) &= \mathbb{E}\left[f(z)\right] , \\ \text{Covariance function:} \qquad \kappa(z, \tilde{z}) &= \mathbb{E}\left[\left(f(z) - \mu(z)\right)\left(f(\tilde{z}) - \mu(\tilde{z})\right)\right] . \end{aligned}$$

- Given a data set $\mathcal{D}$ of n observations: $\mathcal{D} \equiv \left\{(z_i, y_i)|_{i=1,\ldots,n}\right\}$.
- We attempt to reconstruct a function f that describes this data.

We consider a set of training points $\mathbf{Z} = \{z_i\}$.

A function f evaluated at z_i, is a random variable with Gaussian distribution, such that the vector $\mathbf{f} = \{f_i\}$ has a multivariate Gaussian distribution:

$$\mathbf{f} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{K}),$$

where, $\boldsymbol{\mu} = \{\mu(z_i)\}$.

The covariance matrix $\mathbf{K} \equiv \kappa(\mathbf{Z}, \mathbf{Z})$ is given by $[\kappa(\mathbf{Z}, \mathbf{Z})]_{ij} = \kappa(z_i, z_j)$.

Consider a set of test points $\mathbf{Z}^* = \{z_i^*\}$.

The Gaussian vector $\mathbf{f}^*$ of function values at $\mathbf{Z}^*$ with $f_i^* = f(z_i^*)$, is

$$\mathbf{f}^* \sim \mathcal{N}\left(\boldsymbol{\mu}^*, \mathbf{K}^{**}\right).$$

Here $\boldsymbol{\mu}^*$ is the mean of $\mathbf{f}^*$, and $\mathbf{K}^{**} = \kappa(\mathbf{Z}^*, \mathbf{Z}^*)$.

The joint distribution of $\mathbf{f}$ and $\mathbf{f}^*$:

$$\begin{bmatrix} \mathbf{f} \\ \mathbf{f}^* \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \boldsymbol{\mu} \\ \boldsymbol{\mu}^* \end{bmatrix} \begin{bmatrix} \mathbf{K} & \mathbf{K}^* \\ \mathbf{K}^{*\mathsf{T}} & \mathbf{K}^{**} \end{bmatrix} \right),$$

where, $\mathbf{K}^* = \kappa(\mathbf{Z}, \mathbf{Z}^*)$ and $\mathbf{K}^{*\mathsf{T}} = \kappa(\mathbf{Z}^*, \mathbf{Z})$ respectively.

For noisy data $\{z_i, y_i \equiv f(z_i) + \epsilon_i\}$ (with Gaussian noise ϵ_i and variance σ_i^2):

$$\mathbf{y} \sim \mathcal{N}\left(\boldsymbol{\mu}, \mathbf{K} + \sigma_i^2 \mathbf{I}\right),$$

For correlated data: Add $\mathbf{C}$ (covariance matrix of the data) instead of $\sigma_i^2 \mathbf{I}$.

The joint distribution of $\mathbf{y}$ and $\mathbf{f}^*$:

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{f}^* \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \boldsymbol{\mu} \\ \boldsymbol{\mu}^* \end{bmatrix} \begin{bmatrix} \mathbf{K} + \mathbf{C} & \mathbf{K}^* \\ \mathbf{K}^{*\mathsf{T}} & \mathbf{K}^{**} \end{bmatrix}\right),$$

Conditional distribution gives:

$$\mathbf{f}^* | \mathbf{X}^*, \mathbf{X}, \mathbf{y} \sim \mathcal{N}\left(\overline{\boldsymbol{\mu}}^*, \boldsymbol{\Sigma}^*\right),$$
$$\overline{\boldsymbol{\mu}}^* = \boldsymbol{\mu}^* + \mathbf{K}^{*\mathsf{T}} [\mathbf{K} + \mathbf{C}]^{-1} (\mathbf{y} - \boldsymbol{\mu}),$$
$$\boldsymbol{\Sigma}^* = \mathbf{K}^{**} - \mathbf{K}^{*\mathsf{T}} [\mathbf{K} + \mathbf{C}]^{-1} \mathbf{K}^*.$$

$\mathbf{f}^* | \mathbf{X}^*, \mathbf{X}, \mathbf{y} \longrightarrow$ distribution over noise-free predictions.

Include the contribution from noise into our final predictions $\mathbf{y}^*$, by adding $\mathbf{C}$ to $\boldsymbol{\Sigma}^*$:

$$\mathbf{y}^* | \mathbf{X}^*, \mathbf{X}, \mathbf{y} \sim \mathcal{N}\left(\overline{\boldsymbol{\mu}}^*, \boldsymbol{\Sigma}^* + \mathbf{C}\right).$$

Covariance function

- Squared exponential: $\kappa(z,\tilde{z}) = \sigma_f^2 \exp\left[-\frac{(z-\tilde{z})^2}{2l^2}\right]$
- Matérn 9/2: $\kappa(z,\tilde{z}) = \sigma_f^2 \exp\left(\frac{-3|z-\tilde{z}|}{l}\right)\left[1+\frac{3|z-\tilde{z}|}{l}+\frac{27(z-\tilde{z})^2}{7l^2}+\frac{18|z-\tilde{z}|^3}{7l^3}+\frac{27(z-\tilde{z})^4}{35l^4}\right]$
- Cauchy: $\kappa(z,\tilde{z}) = \sigma_f^2\left[\frac{l}{(z-\tilde{z})^2+l^2}\right]$
- Rational quadratic: $\kappa(z,\tilde{z}) = \sigma_f^2\left[1+\frac{(z-\tilde{z})^2}{2\alpha l^2}\right]^{-\alpha}$

$\kappa(z,\tilde{z})$ depends on **hyperparameters** (σ_f, l, α).

The log-marginal likelihood is:

$$\boxed{\ln\mathcal{L} = -\frac{1}{2}(\mathbf{y}-\boldsymbol{\mu})^{\mathrm{T}}[\mathbf{K}+\mathbf{C}]^{-1}\,(\mathbf{y}-\boldsymbol{\mu}) - \frac{1}{2}\ln|\mathbf{K}+\mathbf{C}| - \frac{n}{2}\ln 2\pi}.$$

The hyperparameters σ_f, l and α are obtained by maximizing this log-marginal likelihood.

Observational Data

- BACKGROUND LEVEL $\rightarrow$
  - Cosmic Chronometer Hubble data (CC): $\{z_i, H_i, \sigma_{H_i}\}$
  - Pantheon type Ia supernova distance modulus data (SN): $\{z_i, \mu_i, \sigma_{\mu_i}\}$
  - Baryon Acoustic Oscillation Hubble data (BAO): $\{z_i, H_i, \sigma_{H_i}\}$
- PERTURBATION LEVEL $\rightarrow$
  - Redshift Space Distortion data (RSD): $\{z_i, f\sigma_8(z_i), \sigma_{f\sigma_8}(z_i)\}$
- EFFECT OF H_0 PRIORS $\rightarrow$
  - Planck 2018 H_0 (P18): $\{z = 0, H_0, \sigma_{H_0}\}$
  - Riess 2019 H_0 (R19): $\{z = 0, H_0, \sigma_{H_0}\}$

Theoretical Framework

The reduced Hubble parameter:

$$E(z) = H(z)/H_0.$$

The normalized proper distance:

$$D_p(z) = \int_0^z \frac{dz'}{E(z')}.$$

The normalized transverse comoving distance:

$$D(z) = \frac{1}{\sqrt{|\Omega_{k0}|}} \sin n \left(\sqrt{|\Omega_{k0}|} \int_0^z \frac{dz'}{E(z')} \right),$$

where,

$$\sin nx = \begin{cases} \sinh x & (\Omega_{k0} > 0), \\ x & (\Omega_{k0} = 0), \\ \sin x & (\Omega_{k0} < 0). \end{cases}$$

Reconstruction from Background data

Use GP to reconstruct $H(z)$ from CC and CC + BAO data.

Normalize datasets with the reconstructed H_0, i.e., $H(z=0)$ to obtain $E(z)$.

Compute D_p from reconstructed $H(z)$ via the composite trapezoidal rule[1]:

$$D_p(z) \simeq \frac{1}{2}\sum_{i=0}^{n}(z_{i+1}-z_i)\left[\frac{1}{E(z_{i+1})}+\frac{1}{E(z_i)}\right].$$

$$\text{Calculate: } D\{\Omega_{k0}\}(z) = \begin{cases} \frac{1}{\sqrt{\Omega_{k0}}}\sinh\left[\sqrt{\Omega_{k0}}\; D_p(z)\right] & \Omega_{k0} > 0, \\ D_p(z) & \Omega_{k0} = 0, \\ \frac{1}{\sqrt{-\Omega_{k0}}}\sin\left[\sqrt{-\Omega_{k0}}\; D_p(z)\right] & \Omega_{k0} < 0. \end{cases}$$

Another GP to reconstruct SN-Ia distance modulus $\mu_{\rm SN}(z)$.

[1] R. Holanda, J. Carvalho & J. Alcaniz, JCAP **04**(2013)027

Theoretically distance modulus:

$$\mu = 5 \log \left(\frac{d_L}{\text{Mpc}} \right) + 25.$$

Here, $d_L \rightarrow$ luminosity distance.

$$\text{Compute } \mu_{\text{H}}\{\Omega_{k0}, H_0\} = 5 \log_{10} \left[\frac{c(1+z)D\{\Omega_{k0}\}}{H_0} \right] + 25.$$

$$\text{Estimate the uncertainty: } \sigma_{\mu_{\text{H}}}\{\Omega_{k0}, H_0\} = \frac{5}{\ln 10} \frac{\sigma_D\{\Omega_{k0}\}}{D\{\Omega_{k0}\}}.$$

Minimize:

$$\chi^2\{\Omega_{k0}, H_0\} = \left(\boldsymbol{\mu}_{\text{SN}} - \boldsymbol{\mu}_{\text{H}}\{\Omega_{k0}, H_0\}\right)^{\text{T}} \ \boldsymbol{\Sigma}^{-1} \ \left(\boldsymbol{\mu}_{\text{SN}} - \boldsymbol{\mu}_{\text{H}}\{\Omega_{k0}, H_0\}\right). \tag{3}$$

Markov Chain Monte Carlo (MCMC) analysis: uniform priors $\rightarrow \Omega_{k0} \in [-1, 1]$ and $H_0 \in [50, 100]$.

Effect of H_0 priors:

- P18 prior of $H_0 = 67.27 \pm 0.60$ km Mpc^{-1} s^{-1},
- R19 prior of $H_0 = 74.03 \pm 1.42$ km Mpc^{-1} s^{-1}.

H_0 values are added to the Hubble dataset and considered for MCMC analysis.

Thermodynamic consistency of Ω_{k0} constraints

Assume the universe as a system is bounded by the apparent horizon r_A.

The net matter content is enclosed within a volume defined by a radius not bigger than the horizon[2].

For the FLRW universe with a spatial curvature index k: $r_A = \left(H^2 + \frac{k}{a^2}\right)^{-\frac{1}{2}}$.

Entropy of the horizon: $S_A = 8\pi^2 r_A^2 = \frac{8\pi^2}{H^2+\frac{k}{a^2}}$.

From second law: The total entropy of the system, i.e., $S = S_f + S_A$, should satisfy the relation

$$\frac{dS}{da} \equiv \frac{dS_f}{da} + \frac{dS_A}{da} \geq 0; \; S_f \; \rightarrow \text{ entropy of the fluid describing the observable universe}$$

Assumption [S. Frautschi, Science **217**, 593 (1982)]: $S_A >> S_f$.

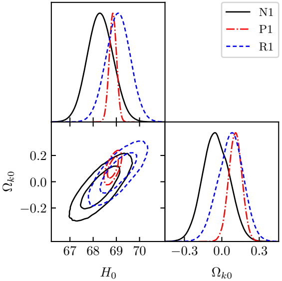
The second law of thermodynamics [P. C. Ferreira and D. Pavón, Universe **2(4)**, 27 (2016)]:

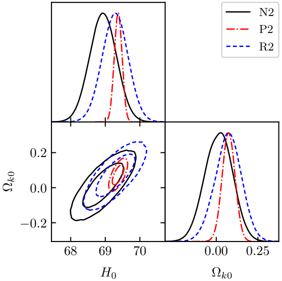
$$\frac{dS_A}{da} \geq 0 \implies H\frac{dH}{da} \leq \frac{k}{a^3} \implies \boxed{1 + q \geq \Omega_k}.$$

[2] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2738 (1977)

Squared Exponential covariance

- Set I: N1 - CC+SN, P1 - CC+SN+P18, R1 - CC+SN+R19.
- Set II: N2 - CC+BAO+SN, P2 - CC+BAO+SN+P18, R2 - CC+BAO+SN+R19.

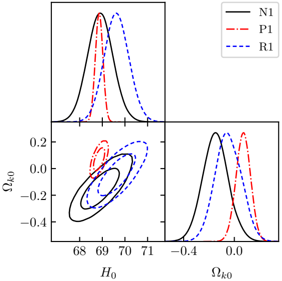


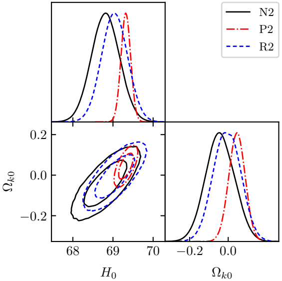


Set	H_0	Ω_{k0}	$1+q_0$
N1	$68.30^{+0.56\ +1.09\ +1.67}_{-0.55\ -1.06\ -1.59}$	$-0.05^{+0.11\ +0.21\ +0.32}_{-0.10\ -0.20\ -0.31}$	$0.49^{+0.06\ +0.12\ +0.18}_{-0.06\ -0.13\ -0.19}$
N2	$68.94^{+0.39\ +0.76\ +1.15}_{-0.39\ -0.75\ -1.14}$	$0.02^{+0.08\ +0.16\ +0.24}_{-0.09\ -0.16\ -0.25}$	$0.42^{+0.03\ +0.06\ +0.09}_{-0.03\ -0.06\ -0.09}$
P1	$68.85^{+0.16\ +0.31\ +0.46}_{-0.16\ -0.31\ -0.46}$	$0.11^{+0.05\ +0.11\ +0.16}_{-0.05\ -0.11\ -0.16}$	$0.46^{+0.06\ +0.12\ +0.18}_{-0.06\ -0.12\ -0.18}$
P2	$69.37^{+0.11\ +0.22\ +0.33}_{-0.11\ -0.22\ -0.33}$	$0.07^{+0.04\ +0.08\ +0.12}_{-0.04\ -0.08\ -0.12}$	$0.39^{+0.03\ +0.06\ +0.10}_{-0.03\ -0.06\ -0.10}$
R1	$69.08^{+0.52\ +1.02\ +1.54}_{-0.52\ -1.03\ -1.55}$	$0.07^{+0.10\ +0.19\ +0.29}_{-0.11\ -0.21\ -0.31}$	$0.45^{+0.06\ +0.12\ +0.18}_{-0.06\ -0.12\ -0.18}$
R2	$69.29^{+0.37\ +0.72\ +1.10}_{-0.37\ -0.73\ -1.10}$	$0.07^{+0.08\ +0.15\ +0.23}_{-0.08\ -0.16\ -0.24}$	$0.40^{+0.03\ +0.06\ +0.09}_{-0.03\ -0.06\ -0.09}$

Matérn $9/2$ covariance

- Set I: N1 - CC+SN, P1 - CC+SN+P18, R1 - CC+SN+R19.
- Set II: N2 - CC+BAO+SN, P2 - CC+BAO+SN+P18, R2 - CC+BAO+SN+R19.

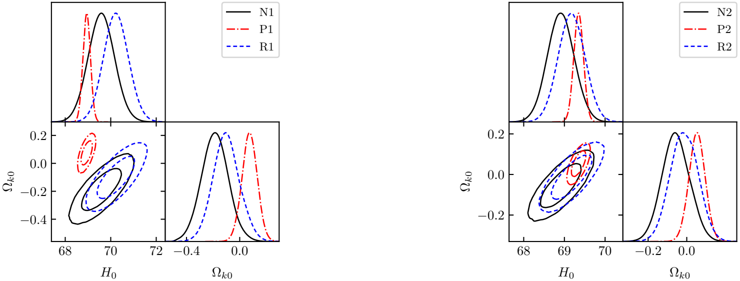




Set	H_0	Ω_{k0}	$1+q_0$
N1	$68.91^{+0.56\ +1.13\ +1.72}_{-0.56\ -1.10\ -1.65}$	$-0.15^{+0.10\ +0.21\ +0.32}_{-0.10\ -0.20\ -0.31}$	$0.50^{+0.05\ +0.11\ +0.15}_{-0.05\ -0.10\ -0.16}$
N2	$68.81^{+0.35\ +0.69\ +1.06}_{-0.35\ -0.68\ -1.03}$	$-0.04^{+0.08\ +0.15\ +0.22}_{-0.07\ -0.14\ -0.22}$	$0.43^{+0.05\ +0.09\ +0.14}_{-0.05\ -0.09\ -0.14}$
P1	$68.86^{+0.17\ +0.33\ +0.49}_{-0.17\ -0.33\ -0.49}$	$0.07^{+0.06\ +0.11\ +0.17}_{-0.06\ -0.11\ -0.17}$	$0.51^{+0.05\ +0.10\ +0.15}_{-0.05\ -0.10\ -0.15}$
P2	$69.32^{+0.12\ +0.24\ +0.36}_{-0.12\ -0.24\ -0.36}$	$0.05^{+0.04\ +0.08\ +0.12}_{-0.04\ -0.08\ -0.13}$	$0.40^{+0.04\ +0.08\ +0.13}_{-0.04\ -0.08\ -0.13}$
R1	$68.63^{+0.55\ +1.09\ +1.64}_{-0.53\ -1.04\ -1.57}$	$-0.05^{+0.11\ +0.21\ +0.31}_{-0.10\ -0.20\ -0.30}$	$0.52^{+0.05\ +0.12\ +0.17}_{-0.05\ -0.11\ -0.16}$
R2	$69.04^{+0.32\ +0.64\ +0.96}_{-0.32\ -0.63\ -0.95}$	$-0.01^{+0.07\ +0.14\ +0.21}_{-0.07\ -0.14\ -0.21}$	$0.43^{+0.04\ +0.08\ +0.12}_{-0.04\ -0.08\ -0.12}$

Cauchy covariance

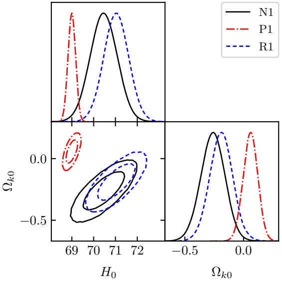
- Set I: N1 - CC+SN, P1 - CC+SN+P18, R1 - CC+SN+R19.
- Set II: N2 - CC+BAO+SN, P2 - CC+BAO+SN+P18, R2 - CC+BAO+SN+R19.

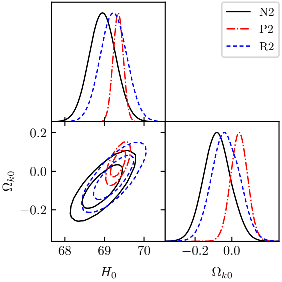


Set	H_0	Ω_{k0}	$1+q_0$
N1	$69.59^{+0.58\ +1.15\ +1.77}_{-0.57\ -1.13\ -1.68}$	$-0.19^{+0.10\ +0.20\ +0.31}_{-0.10\ -0.20\ -0.29}$	$0.41^{+0.07\ +0.13\ +0.20}_{-0.07\ -0.13\ -0.21}$
N2	$68.91^{+0.33\ +0.66\ +0.99}_{-0.33\ -0.64\ -0.97}$	$-0.06^{+0.07\ +0.14\ +0.21}_{-0.07\ -0.14\ -0.21}$	$0.41^{+0.04\ +0.07\ +0.11}_{-0.04\ -0.07\ -0.11}$
P1	$68.94^{+0.16\ +0.32\ +0.49}_{-0.16\ -0.32\ -0.49}$	$0.07^{+0.06\ +0.11\ +0.17}_{-0.06\ -0.12\ -0.17}$	$0.45^{+0.07\ +0.13\ +0.19}_{-0.07\ -0.14\ -0.19}$
P2	$69.35^{+0.12\ +0.23\ +0.35}_{-0.12\ -0.23\ -0.35}$	$0.05^{+0.04\ +0.08\ +0.12}_{-0.04\ -0.08\ -0.13}$	$0.38^{+0.04\ +0.07\ +0.11}_{-0.04\ -0.07\ -0.11}$
R1	$70.23^{+0.55\ +1.09\ +1.67}_{-0.54\ -1.06\ -1.59}$	$-0.10^{+0.10\ +0.20\ +0.30}_{-0.10\ -0.19\ -0.29}$	$0.38^{+0.07\ +0.15\ +0.21}_{-0.07\ -0.14\ -0.20}$
R2	$69.18^{+0.33\ +0.64\ +0.96}_{-0.32\ -0.63\ -0.95}$	$-0.01^{+0.07\ +0.14\ +0.21}_{-0.07\ -0.14\ -0.21}$	$0.39^{+0.04\ +0.07\ +0.11}_{-0.04\ -0.07\ -0.11}$

Rational quadratic covariance

- Set I: N1 - CC+SN, P1 - CC+SN+P18, R1 - CC+SN+R19.
- Set II: N2 - CC+BAO+SN, P2 - CC+BAO+SN+P18, R2 - CC+BAO+SN+R19.





Set	H_0	Ω_{k0}	$1 + q_0$
N1	$70.46^{+0.62\ +1.22\ +1.87}_{-0.62\ -1.21\ -1.83}$	$-0.26^{+0.10\ +0.20\ +0.31}_{-0.10\ -0.21\ -0.31}$	$0.37^{+0.07\ +0.14\ +0.20}_{-0.07\ -0.13\ -0.20}$
N2	$68.95^{+0.34\ +0.67\ +1.02}_{-0.33\ -0.65\ -0.99}$	$-0.08^{+0.08\ +0.15\ +0.22}_{-0.07\ -0.14\ -0.22}$	$0.41^{+0.04\ +0.07\ +0.11}_{-0.04\ -0.08\ -0.11}$
P1	$68.99^{+0.17\ +0.34\ +0.52}_{-0.17\ -0.34\ -0.52}$	$0.06^{+0.06\ +0.12\ +0.18}_{-0.06\ -0.12\ -0.19}$	$0.45^{+0.07\ +0.13\ +0.20}_{-0.07\ -0.13\ -0.20}$
P2	$69.35^{+0.13\ +0.25\ +0.38}_{-0.13\ -0.25\ -0.38}$	$0.04^{+0.05\ +0.09\ +0.14}_{-0.05\ -0.09\ -0.14}$	$0.38^{+0.04\ +0.08\ +0.12}_{-0.04\ -0.08\ -0.12}$
R1	$71.03^{+0.57\ +1.22\ +1.70}_{-0.56\ -1.11\ -1.68}$	$-0.19^{+0.10\ +0.20\ +0.30}_{-0.10\ -0.20\ -0.30}$	$0.37^{+0.06\ +0.12\ +0.17}_{-0.06\ -0.12\ -0.17}$
R2	$69.23^{+0.34\ +0.66\ +1.00}_{-0.33\ -0.64\ -0.97}$	$-0.03^{+0.08\ +0.15\ +0.22}_{-0.07\ -0.14\ -0.22}$	$0.39^{+0.04\ +0.07\ +0.11}_{-0.04\ -0.07\ -0.11}$

Reconstruction along with Perturbation data

The evolution of matter density contrast is, $\delta = \frac{\delta\rho_m}{\rho_m}$.

In the linearized approximation, δ obeys the following second order differential equation:

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G_{\text{eff}}\rho_m\delta = 0,$$

where $\rho_m \rightarrow$ background matter density, $\delta\rho_m \rightarrow$ first-order matter perturbation

The RSD data measure $f\sigma_8 \rightarrow$ **growth rate of structure**:

$$\begin{aligned} f\sigma_8(z) &= -\frac{\sigma_8(z=0)}{\delta(z=0)}(1+z)\delta', \\ f(z) &= -(1+z)\frac{\mathrm{d}\ln\delta}{\mathrm{d}z} = -(1+z)\frac{\delta'}{\delta}, \\ \sigma_8(z) &= \sigma_8(z=0)\frac{\delta(z)}{\delta(z=0)}, \end{aligned}$$

where, $f \rightarrow$ **growth rate**,

$\sigma_8 \rightarrow$ **root-mean-square mass fluctuation within a sphere of radius** $8h^{-1}$ **Mpc**.

Approximate solution: [Y. Gong, M. Ishak & A. Wang, Phys. Rev. D **80** 023002 (2009)]

$$f(z) = \Omega_m^{\gamma} + \left(\gamma - \frac{4}{7}\right)\Omega_k.$$

Here, $\Omega_m = \frac{\Omega_{m0}(1+z)^3}{E^2(z)}$, $\Omega_k = \frac{\Omega_{k0}(1+z)^2}{E^2(z)}$ and $\gamma \rightarrow$ **growth index of perturbations**.

Theoretical $f\sigma_8$:

$$f\sigma_8^{\rm theo}(z) = \sigma_{8,0}\left[\Omega_m^{\gamma} + \left(\gamma - \frac{4}{7}\right)\Omega_k\right]\exp\left\{\int_0^z -\frac{\left[\Omega_m^{\gamma} + \left(\gamma - \frac{4}{7}\right)\Omega_k\right]}{1+z'}dz'\right\}.$$

Minimize $\chi^2_{\rm RSD}$:

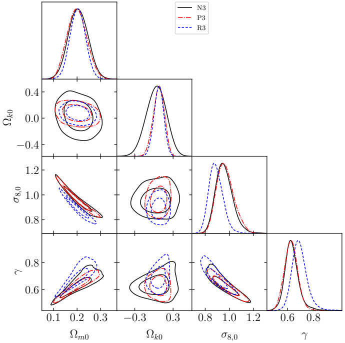
$$\chi^2 = \Delta\mathbf{V}^{\mathbf{T}}\ \mathbf{Cov}^{-1}\ \Delta\mathbf{V},$$
$$\Delta V_i = f\sigma_8^{\rm obs}(z_i) - f\sigma_8^{\rm theo}(z_i),$$
$$\mathbf{Cov} = \mathbf{Cov}^{\rm obs} + \mathbf{Cov}^{\rm theo}.$$

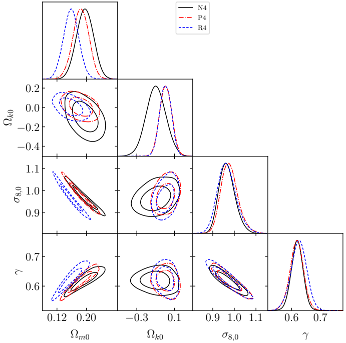
$\mathbf{Cov}^{\rm obs} \rightarrow$ covariance matrix of $f\sigma_8^{\rm obs}$, $\mathbf{Cov}^{\rm theo} \rightarrow$ covariance matrix of $f\sigma_8^{\rm theo}$.

Ω_{k0} constraints obtained by MCMC analysis.

Squared Exponential covariance

- Set III: N3 - CC+SN+RSD, P3 - CC+SN+P18+RSD, R3 - CC+SN+R19+RSD.
- Set IV: N4 - CC+BAO+SN+RSD, P4 - CC+BAO+SN+P18+RSD, R4 - CC+BAO+SN+R19+RSD.

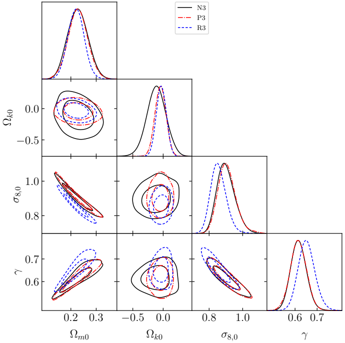


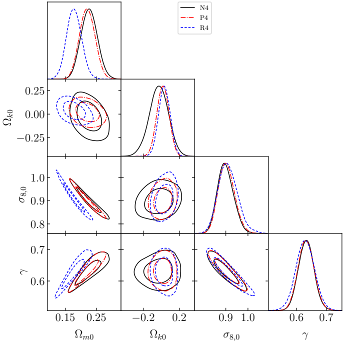


Set	Ω_{m0}	Ω_{k0}	$\sigma_{8,0}$	γ
N3	$0.204^{+0.042\ +0.082\ +0.126}_{-0.041\ -0.079\ -0.121}$	$0.040^{+0.152\ +0.285\ +0.419}_{-0.161\ -0.313\ -0.483}$	$0.952^{+0.074\ +0.163\ +0.296}_{-0.063\ -0.116\ -0.171}$	$0.629^{+0.053\ +0.139\ +0.311}_{-0.045\ -0.094\ -0.148}$
N4	$0.196^{+0.023\ +0.044\ +0.068}_{-0.023\ -0.045\ -0.065}$	$-0.097^{+0.105\ +0.205\ +0.299}_{-0.102\ -0.202\ -0.314}$	$0.964^{+0.039\ +0.082\ +0.120}_{-0.034\ -0.063\ -0.090}$	$0.619^{+0.020\ +0.040\ +0.066}_{-0.020\ -0.041\ -0.079}$
P3	$0.199^{+0.042\ +0.083\ +0.125}_{-0.041\ -0.075\ -0.099}$	$0.077^{+0.084\ +0.159\ +0.212}_{-0.091\ -0.178\ -0.265}$	$0.961^{+0.077\ +0.159\ +0.232}_{-0.065\ -0.118\ -0.164}$	$0.626^{+0.049\ +0.102\ +0.174}_{-0.049\ -0.095\ -0.132}$
P4	$0.185^{+0.021\ +0.041\ +0.066}_{-0.021\ -0.043\ -0.063}$	$0.007^{+0.065\ +0.127\ +0.218}_{-0.063\ -0.122\ -0.186}$	$0.976^{+0.040\ +0.089\ +0.141}_{-0.036\ -0.067\ -0.100}$	$0.618^{+0.023\ +0.047\ +0.103}_{-0.023\ -0.050\ -0.078}$
R3	$0.203^{+0.032\ +0.064\ +0.096}_{-0.033\ -0.064\ -0.108}$	$0.078^{+0.073\ +0.144\ +0.212}_{-0.077\ -0.155\ -0.232}$	$0.885^{+0.063\ +0.132\ +0.250}_{-0.052\ -0.098\ -0.144}$	$0.688^{+0.058\ +0.139\ +0.278}_{-0.050\ -0.098\ -0.156}$
R4	$0.159^{+0.020\ +0.042\ +0.077}_{-0.021\ -0.041\ -0.056}$	$0.005^{+0.061\ +0.115\ +0.171}_{-0.060\ -0.119\ -0.228}$	$0.963^{+0.045\ +0.097\ +0.148}_{-0.040\ -0.076\ -0.117}$	$0.625^{+0.026\ +0.052\ +0.079}_{-0.027\ -0.056\ -0.081}$

Matérn $9/2$ covariance

- Set III: N3 - CC+SN+RSD, P3 - CC+SN+P18+RSD, R3 - CC+SN+R19+RSD.
- Set IV: N4 - CC+BAO+SN+RSD, P4 - CC+BAO+SN+P18+RSD, R4 - CC+BAO+SN+R19+RSD.

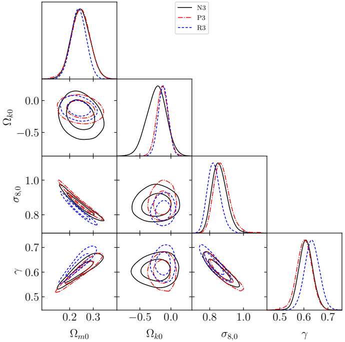


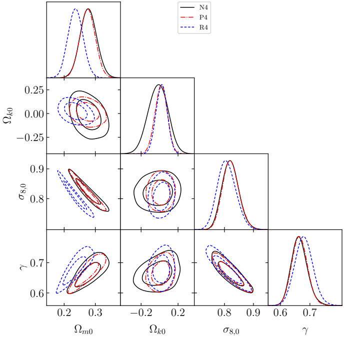


Set	Ω_{m0}	Ω_{k0}	$\sigma_{8,0}$	γ
N3	$0.227^{+0.041\ +0.079\ +0.117}_{-0.040\ -0.074\ -0.105}$	$-0.110^{+0.155\ +0.307\ +0.445}_{-0.155\ -0.312\ -0.498}$	$0.897^{+0.057\ +0.118\ +0.183}_{-0.048\ -0.089\ -0.125}$	$0.615^{+0.036\ +0.072\ +0.118}_{-0.034\ -0.066\ -0.097}$
N4	$0.227^{+0.027\ +0.053\ +0.079}_{-0.026\ -0.049\ -0.074}$	$-0.026^{+0.104\ +0.202\ +0.297}_{-0.104\ -0.215\ -0.322}$	$0.897^{+0.039\ +0.076\ +0.125}_{-0.034\ -0.064\ -0.090}$	$0.633^{+0.025\ +0.052\ +0.082}_{-0.026\ -0.050\ -0.078}$
P3	$0.228^{+0.038\ +0.077\ +0.114}_{-0.038\ -0.077\ -0.108}$	$-0.044^{+0.089\ +0.175\ +0.264}_{-0.090\ -0.176\ -0.270}$	$0.903^{+0.055\ +0.125\ +0.195}_{-0.048\ -0.091\ -0.127}$	$0.615^{+0.037\ +0.073\ +0.114}_{-0.035\ -0.073\ -0.107}$
P4	$0.221^{+0.025\ +0.049\ +0.076}_{-0.024\ -0.047\ -0.066}$	$0.018^{+0.067\ +0.128\ +0.185}_{-0.067\ -0.131\ -0.196}$	$0.902^{+0.038\ +0.077\ +0.115}_{-0.035\ -0.065\ -0.095}$	$0.632^{+0.027\ +0.054\ +0.083}_{-0.026\ -0.051\ -0.072}$
R3	$0.220^{+0.032\ +0.064\ +0.100}_{-0.032\ -0.062\ -0.093}$	$-0.031^{+0.083\ +0.159\ +0.235}_{-0.081\ -0.159\ -0.242}$	$0.853^{+0.048\ +0.102\ +0.171}_{-0.042\ -0.079\ -0.115}$	$0.651^{+0.039\ +0.080\ +0.125}_{-0.036\ -0.070\ -0.107}$
R4	$0.178^{+0.024\ +0.049\ +0.076}_{-0.023\ -0.045\ -0.065}$	$0.034^{+0.065\ +0.124\ +0.177}_{-0.063\ -0.123\ -0.188}$	$0.907^{+0.046\ +0.096\ +0.147}_{-0.040\ -0.076\ -0.110}$	$0.629^{+0.031\ +0.061\ +0.093}_{-0.030\ -0.061\ -0.094}$

Cauchy covariance

- Set III: N3 - CC+SN+RSD, P3 - CC+SN+P18+RSD, R3 - CC+SN+R19+RSD.
- Set IV: N4 - CC+BAO+SN+RSD, P4 - CC+BAO+SN+P18+RSD, R4 - CC+BAO+SN+R19+RSD.

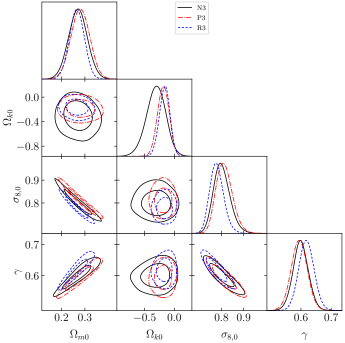


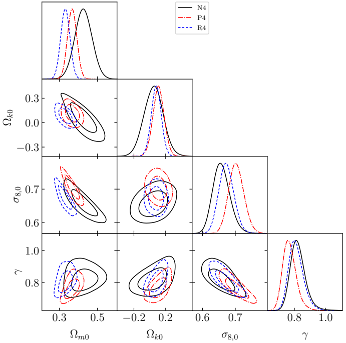


Set	Ω_{m0}	Ω_{k0}	$\sigma_{8,0}$	γ
N3	$0.245^{+0.040\ +0.078\ +0.120}_{-0.038\ -0.073\ -0.115}$	$-0.227^{+0.146\ +0.291\ +0.426}_{-0.164\ -0.318\ -0.494}$	$0.855^{+0.047\ +0.099\ +0.169}_{-0.041\ -0.075\ -0.107}$	$0.608^{+0.029\ +0.059\ +0.089}_{-0.029\ -0.056\ -0.091}$
N4	$0.278^{+0.027\ +0.053\ +0.080}_{-0.026\ -0.050\ -0.076}$	$-0.015^{+0.104\ +0.198\ +0.295}_{-0.108\ -0.208\ -0.314}$	$0.820^{+0.029\ +0.061\ +0.097}_{-0.028\ -0.052\ -0.076}$	$0.663^{+0.028\ +0.060\ +0.099}_{-0.026\ -0.048\ -0.071}$
P3	$0.246^{+0.039\ +0.076\ +0.115}_{-0.039\ -0.082\ -0.132}$	$-0.137^{+0.092\ +0.183\ +0.285}_{-0.091\ -0.180\ -0.273}$	$0.868^{+0.049\ +0.114\ +0.229}_{-0.042\ -0.077\ -0.110}$	$0.602^{+0.031\ +0.062\ +0.098}_{-0.031\ -0.065\ -0.127}$
P4	$0.275^{+0.024\ +0.048\ +0.073}_{-0.024\ -0.047\ -0.071}$	$0.017^{+0.063\ +0.124\ +0.183}_{-0.067\ -0.130\ -0.196}$	$0.821^{+0.029\ +0.060\ +0.095}_{-0.027\ -0.051\ -0.075}$	$0.665^{+0.027\ +0.055\ +0.086}_{-0.026\ -0.050\ -0.074}$
R3	$0.238^{+0.032\ +0.063\ +0.092}_{-0.032\ -0.063\ -0.092}$	$-0.123^{+0.081\ +0.159\ +0.246}_{-0.080\ -0.161\ -0.243}$	$0.823^{+0.041\ +0.087\ +0.136}_{-0.036\ -0.067\ -0.096}$	$0.633^{+0.031\ +0.061\ +0.097}_{-0.030\ -0.059\ -0.090}$
R4	$0.236^{+0.024\ +0.048\ +0.072}_{-0.024\ -0.045\ -0.072}$	$0.022^{+0.059\ +0.115\ +0.169}_{-0.061\ -0.119\ -0.179}$	$0.807^{+0.033\ +0.067\ +0.112}_{-0.030\ -0.057\ -0.083}$	$0.679^{+0.031\ +0.063\ +0.102}_{-0.030\ -0.058\ -0.092}$

Rational quadratic covariance

- Set III: N3 - CC+SN+RSD, P3 - CC+SN+P18+RSD, R3 - CC+SN+R19+RSD.
- Set IV: N4 - CC+BAO+SN+RSD, P4 - CC+BAO+SN+P18+RSD, R4 - CC+BAO+SN+R19+RSD.





Set	Ω_{m0}	Ω_{k0}	$\sigma_{8,0}$	γ
N3	$0.271^{+0.039\ +0.079\ +0.120}_{-0.040\ -0.079\ -0.118}$	$-0.305^{+0.155\ +0.298\ +0.453}_{-0.163\ -0.333\ -0.509}$	$0.799^{+0.039\ +0.083\ +0.133}_{-0.034\ -0.064\ -0.094}$	$0.599^{+0.025\ +0.051\ +0.096}_{-0.024\ -0.049\ -0.075}$
N4	$0.426^{+0.045\ +0.090\ +0.135}_{-0.045\ -0.090\ -0.169}$	$0.049^{+0.117\ +0.236\ +0.375}_{-0.121\ -0.235\ -0.351}$	$0.657^{+0.027\ +0.057\ +0.095}_{-0.024\ -0.045\ -0.066}$	$0.819^{+0.050\ +0.118\ +0.239}_{-0.042\ -0.081\ -0.119}$
P3	$0.280^{+0.040\ +0.078\ +0.119}_{-0.040\ -0.078\ -0.124}$	$-0.186^{+0.096\ +0.184\ +0.269}_{-0.098\ -0.191\ -0.292}$	$0.811^{+0.039\ +0.082\ +0.141}_{-0.035\ -0.063\ -0.092}$	$0.594^{+0.026\ +0.053\ +0.082}_{-0.026\ -0.052\ -0.082}$
P4	$0.367^{+0.024\ +0.048\ +0.073}_{-0.024\ -0.047\ -0.072}$	$0.110^{+0.070\ +0.140\ +0.208}_{-0.070\ -0.140\ -0.218}$	$0.703^{+0.024\ +0.048\ +0.074}_{-0.022\ -0.043\ -0.063}$	$0.766^{+0.045\ +0.101\ +0.167}_{-0.038\ -0.071\ -0.103}$
R3	$0.265^{+0.032\ +0.062\ +0.097}_{-0.032\ -0.063\ -0.092}$	$-0.162^{+0.085\ +0.165\ +0.250}_{-0.090\ -0.179\ -0.267}$	$0.777^{+0.033\ +0.069\ +0.107}_{-0.030\ -0.055\ -0.083}$	$0.618^{+0.025\ +0.050\ +0.080}_{-0.025\ -0.049\ -0.073}$
R4	$0.331^{+0.024\ +0.047\ +0.070}_{-0.024\ -0.047\ -0.069}$	$0.082^{+0.062\ +0.124\ +0.182}_{-0.062\ -0.124\ -0.185}$	$0.672^{+0.023\ +0.047\ +0.075}_{-0.022\ -0.042\ -0.060}$	$0.807^{+0.050\ +0.109\ +0.170}_{-0.041\ -0.076\ -0.110}$

Summary & Conclusion

- Choice of covariance function affects the results.
- Inclusion of the BAO Hubble data results in tighter constraints on Ω_{k0}.
- Constraints on Ω_{k0} quite comfortably satisfies the thermodynamic requirement.
- Possibility for the existence of a non-zero spatial curvature at the present epoch.
- The estimated sign of k depends on the strategies for measuring H_0 to some extent.
- Spatially flat universe is mostly included at 2σ CL.
- Present state of affairs is quite consistent with $k = 0$.

Thank You