Non-linear density-velocity evolution in *f(R)* theories using phase space dynamics

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References: 1. S. N-G and Sandip Chowdhury, [arXiv:2110.05121]
MNRAS, accepted Jan 2022
2. S.N-G, MNRAS 428 (2013) [arXiv:1207.2294]

IIT Madras, 5th March 2022

Motivation: Need to modify gravity

- Observations of distant supernovae have indicated that the Universe is accelerating. If consisted only of matter (dark or baryonic) *and* followed Einstein's equations it should decelerate.

- In the standard cosmological model, known as the ΛCDM model, dark energy is attributed to the cosmological constant Λ. This model fits a large range of observations fairly well.

Problems:

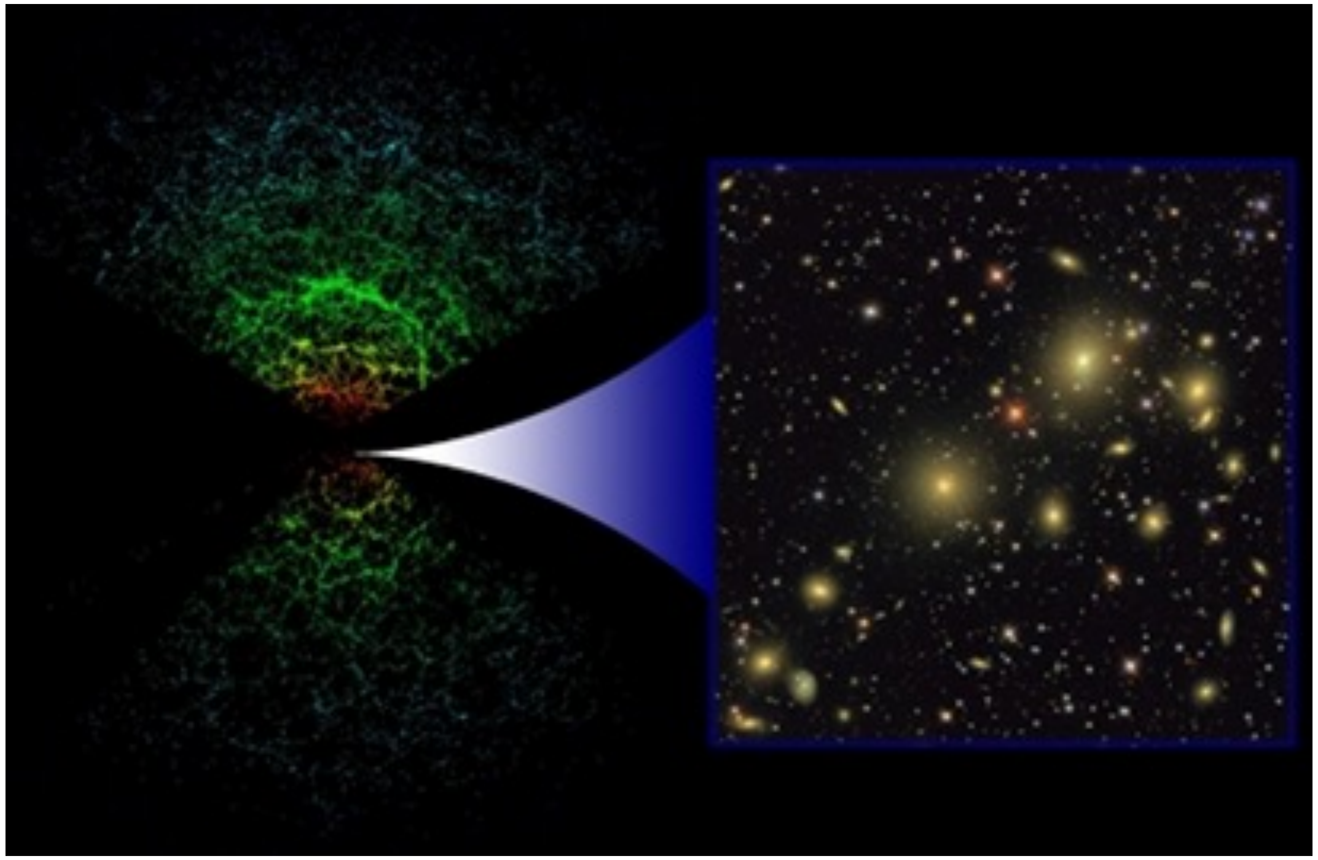
- Mismatch between the theoretical and observational estimate of Λ.
- Hubble tension and σ_8 tension

2 avenues

- Other dark energy models.
- Modify gravity: assume that the equations are wrong.

Reviews: Sotiriou & Faraoni 2010; De Felice & Tsujikawa 2010; Clifton et al. 2012; Joyce et al 2015, 2016, Nojiri et al 2017

Large Scale Structure



Fractional Overdensity

$$\delta = \frac{\rho - \bar{\rho}}{\bar{\rho}}$$

Peculiar velocity

$$\mathbf{v} = \dot{\mathbf{r}} - H\mathbf{r}$$

Linear Relation

$$\Theta \approx -\Omega_m^{\gamma}\delta$$

where $\Theta = \frac{\nabla_r \cdot \mathbf{v}}{H}$

Figure Credit: http://sci.esa.int/planck/47693-large-scale-structure-in-the-universe/

- $\gamma = 0.6$ for pure matter universe Peebles 1976
- Extensively used to get bias independent measures of mass or constrain Ω_m Dekel 1994, Strauss & Wilick 1995
- Extended to dark energy models Lahav et al 1991, Wang & Steinhart 1998, Linder 2005
- γ is sensitive to the cosmological model $\gamma = 0.55$ for ΛCDM, $\gamma = 0.68$ DGP model Linder & Cahn 2007
- Most current and upcoming surveys such as EUCLID, DES, VIPERS, SDSS etc. aim to constrain the growth rate e.g., Guzzo et al. 2008, Majerotto et al. 2012, Alam et al 2020, Perenon et al (2020)

Plan of the talk

- Background cosmology
  -differences between *f(R)* and ΛCDM - presence of oscillations
- *Scalar* Perturbations in *f(R)* models
  - equations and solutions.
- Density velocity dynamics
  - phasespace analysis in ΛCDM (based on S. N-G. MNRAS 2013)
  - phasespace analysis in *f(R)*

Background cosmology

f(R) theories of modified gravity

ΛCDM	*f(R)*
Action	Modified Action
$\frac{1}{16\pi G}\int d^4x\{R-2\Lambda\}+S_m$	$\frac{1}{16\pi G}\int d^4x\{R+f(R)\}+S_m$
Background FRW Metric	Background FRW Metric
$ds^2=-dt^2+a^2(dx^2+dy^2+dz^2)$	$ds^2=-dt^2+a^2(dx^2+dy^2+dz^2)$
Friedmann Equation	Friedmann Equation (modified)
$\frac{\ddot{a}}{a}=-\frac{H_0^2}{2}\left(\frac{\Omega_{m,0}a_0^3}{a^3}-2\Omega_{\Lambda,0}\right)$	$H^2-f_R(HH'+H^2)+\frac{1}{6}f+H^2f_{RR}R'=H^2\Omega_m$ with $H=\frac{\dot{a}}{a}$ and $R=12H^2+6HH'$
Second order Solve given a and $\dot{a}$ at some initial time.	Fourth order (prime is derivative w.r.t. *ln a*) **Method I :** *Designer Approach* Assume *a(t)* is given by ΛCDM and solve for *f(t)* **Method II**: Choose *f(R)* to be a closed form expression

The Hu-Sawicki model

$$f(R) = -m^2 \frac{c_1 (R/m^2)^n}{c_2 (R/m^2)^n + 1}$$

$$m^2 = H_0^2 \Omega_{m,0}$$

Hu & Sawicki 2007

c_1, c_2 and n are three parameters:
One is fixed by demanding ΛCDM at early epochs. Leaves two free parameters. Choose n=1 and the third parameter is fixed by a choice for f_{R0}. Solar system constraints $|f_{R0}| \lesssim 10^{-6}$

Define $y_H = \frac{H^2}{m^2} - a^{-3}$ and $y_R = \frac{R}{m^2} - 3a^{-3}$

Friedmann equation

$$y'_H = \frac{1}{3} y_R - 4 y_H$$

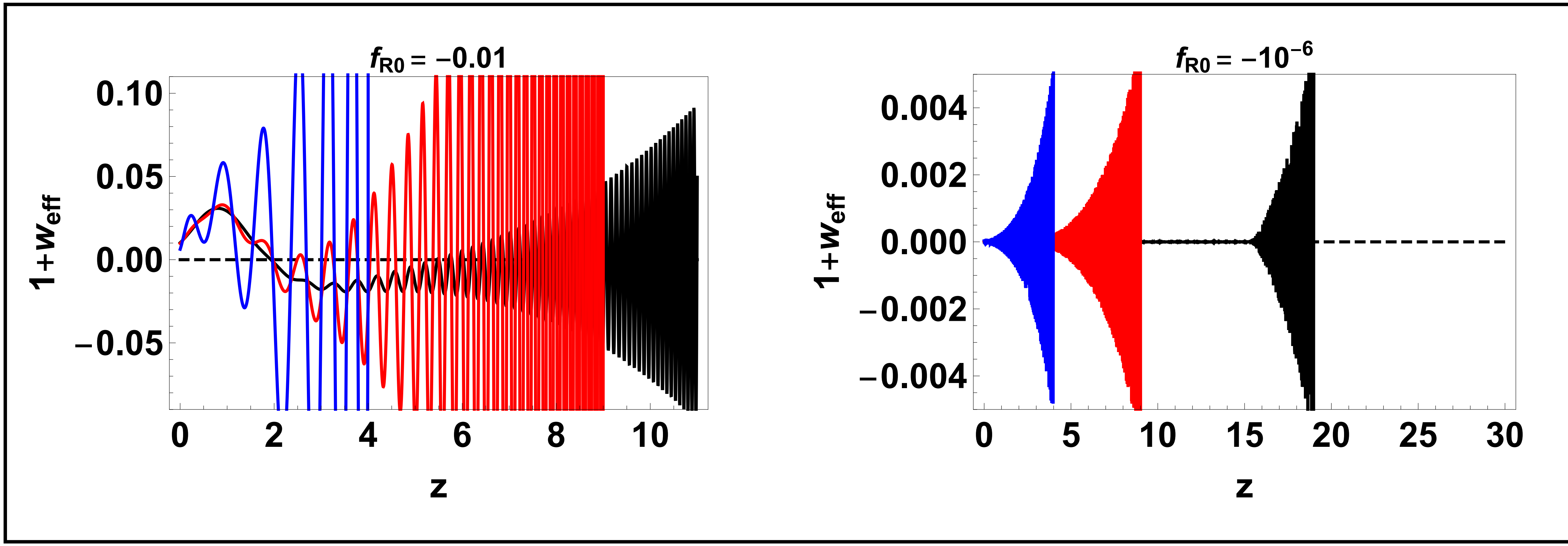
$$y'_R = \frac{9}{a^3} - \frac{1}{\tilde{f}_{\tilde{R}\tilde{R}}(y_H + a^{-3})} \left[y_H - \tilde{f}_{\tilde{R}} \left(\frac{y_R}{6} - y_H - \frac{1}{2a^3} \right) + \frac{\tilde{f}}{6} \right]$$

$$y_{H,\Lambda CDM} = \frac{\Omega_{\Lambda,0}}{\Omega_{m,0}} \qquad y_{R,\Lambda CDM} = \frac{12\Omega_{\Lambda,0}}{\Omega_{m,0}}$$

Q : Are the resulting equations for $H(a)$ consistent with the ΛCDM solutions ? Is the equation of state close to -1 ?

$$1 + w_{eff}(z) = -\frac{1}{3} \frac{y'_H}{y_H}$$

The Hu-Sawicki model



A: Depends on the starting epoch and on the value of f_{R0}

Black: $z_{init} = 4$
Red: $z_{init} = 9$
Blue: $z_{init} = 19$

$$y_H = \frac{H^2}{m^2} - a^{-3}$$

$$1 + w_{eff}(z) = -\frac{1}{3}\frac{y'_H}{y_H}$$

The frequency of oscillations is higher for higher starting epoch and lower $|f_{R0}|$
Amplitude of oscillations lower for smaller values of $|f_{R0}|$
Issue: High frequency oscillations make the system numerically stiff. Evolution from a=0.001 to today.

Aside - Dynamical systems

Autonomous	Non-autonomous
$\dot{x} = f(x, y)$ $\dot{y} = g(x, y)$	$\dot{x} = f(x, y, t)$ $\dot{y} = g(x, y, t)$

- Linear: f and g are linear functions of x and y
- Non-linear: f and g are non-linear of x and y

Linear Autonomous dynamical system	$\dot{x} = a_{11}x + a_{12}y$ $\dot{y} = a_{21}x + a_{22}y$

For linear autonomous systems, eigenvalues of the matrix of coefficients give information about the global solution.

- Positive, real implies a growing solution.
- Negative real, implies a decaying solution.
- Complex implies oscillatory.

For non-linear autonomous systems, linearising gives information about the local solution.

Not quite valid for non-autonomous systems i.e., when the coefficients vary with time. However, the existence of complex eigenvalues can signal oscillations.

Eigenvalue Analysis

Define $y_H = \frac{H^2}{m^2} - a^{-3}$ and $y_R = \frac{R}{m^2} - 3a^{-3}$

Hu & Sawicki 2007

$$y'_H = \frac{1}{3} y_R - 4 y_H$$

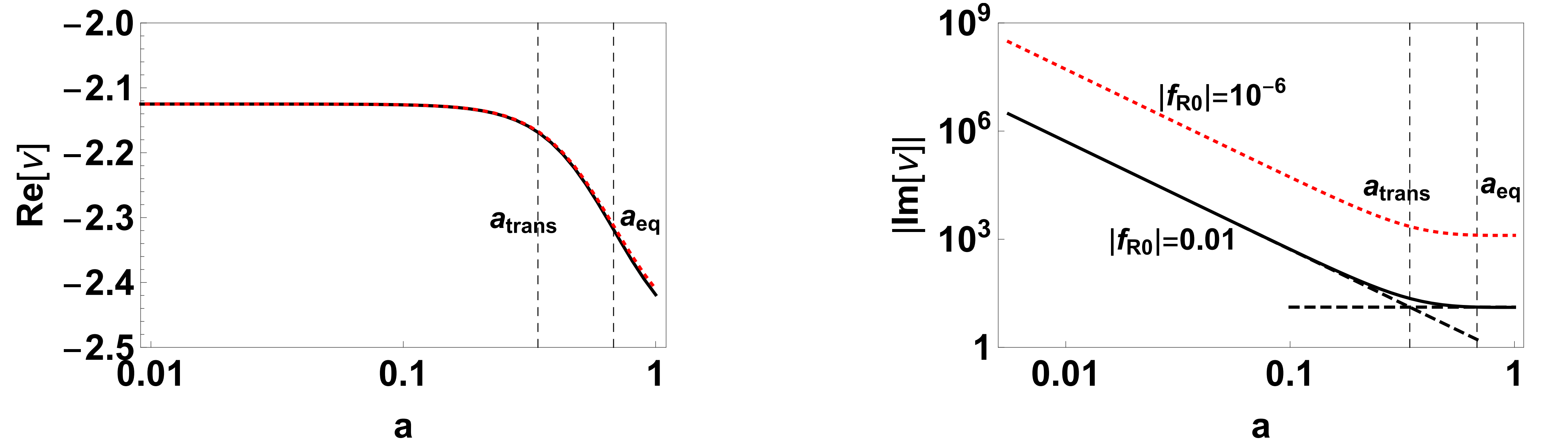
$$y'_R = \frac{9}{a^3} - \frac{1}{\tilde{f}_{\tilde{R}\tilde{R}}(y_H + a^{-3})} \left[y_H - \tilde{f}_{\tilde{R}} \left(\frac{y_R}{6} - y_H - \frac{1}{2a^3} \right) + \frac{\tilde{f}}{6} \right]$$

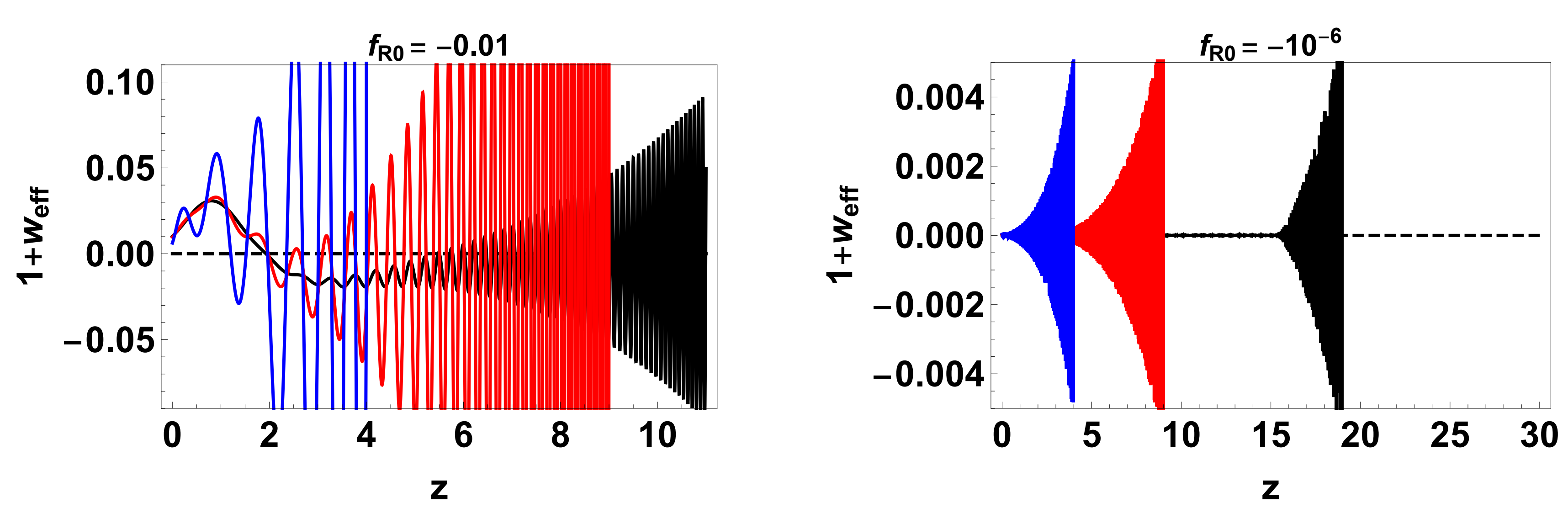
$$1 + w_{eff}(z) = -\frac{1}{3} \frac{y'_H}{y_H}$$

Non-linear non-autonomous system.
Linearize around the instantaneous solution. In practice, we used the ΛCDM solution

$$y_{H,\Lambda CDM} = \frac{\Omega_{\Lambda,0}}{\Omega_{m,0}} \qquad y_{R,\Lambda CDM} = \frac{12\Omega_{\Lambda,0}}{\Omega_{m,0}}$$

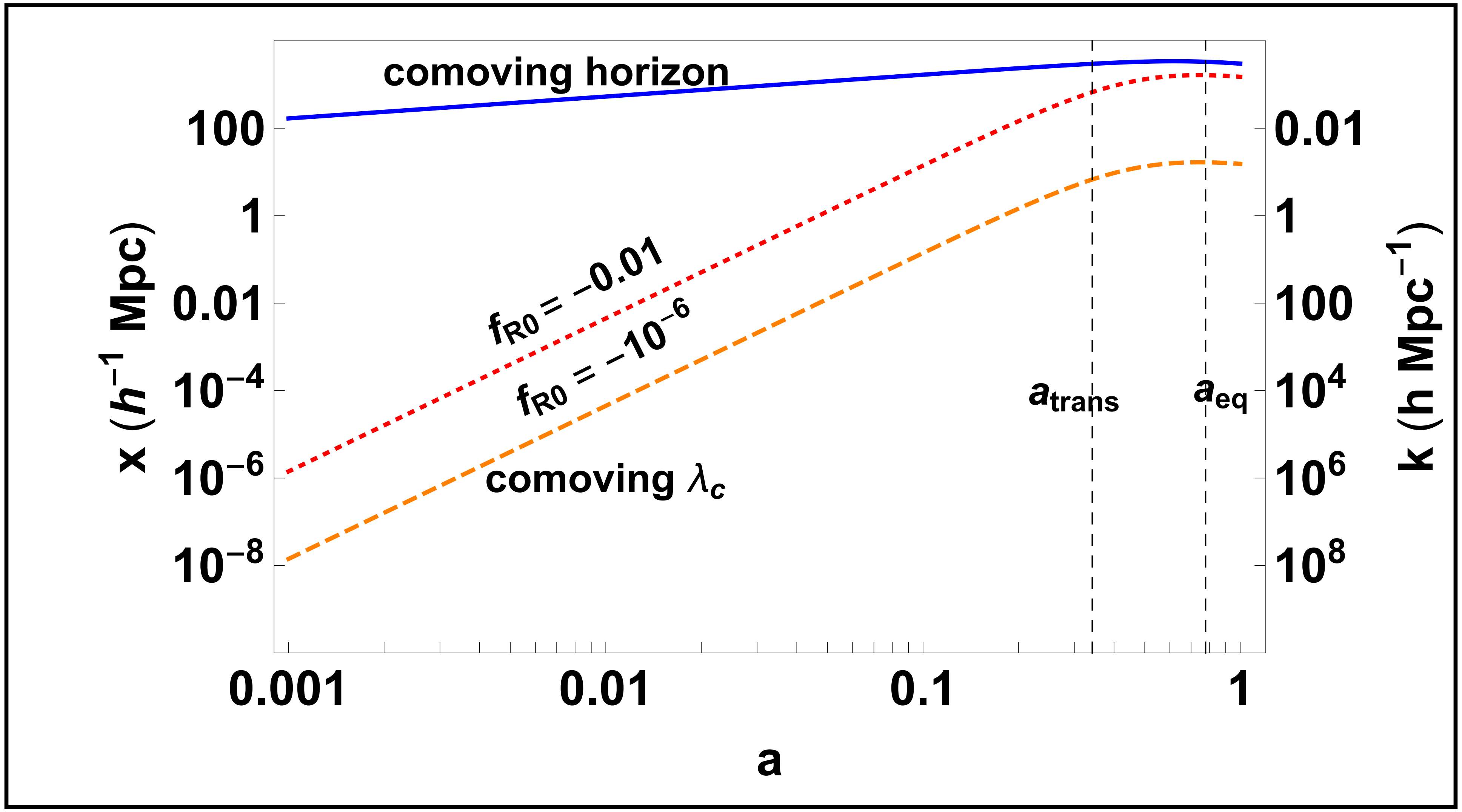
Eigenvalue analysis of the Hu-Sawicki model





This problem also exists for perturbations. The perturbed (non-linear) system is not amenable to eigenvalue analysis, however, we will take insights from this analysis for that system too.
Strategy: assume GR till the epoch a_{switch} after which evolution switches to *f(R)*.
Choose $a_{switch} \sim a_{trans}$.

Compton Wavelength



Compton Wavelength of the associated scalar field

$$\lambda_c = \frac{c\sqrt{3\tilde{f}_{\tilde{R}\tilde{R}}}}{H_0\sqrt{\Omega_{m,0}}}$$

Comoving Compton Wavelength

$$x_C = \frac{\lambda_C}{a}$$

Effect of modification appears on scales smaller than the Compton wavelength

Perturbations

Scalar perturbations in the *f(R)* model

ΛCDM	***f(R)***
Background FRW Metric	Perturbed FRW Metric
$ds^2 = -a^2(1+2\Psi)d\tau^2 + a^2(1-2\Phi)(dx^2+dy^2+dz^2)\,.$	$ds^2 = -a^2(1+2\Psi)d\tau^2 + a^2(1-2\Phi)(dx^2+dy^2+dz^2)\,.$
Sub-horizon Approximation	Quasi-static approximation
1. Length scales are much smaller than the horizon	1. Length scales are much smaller than the horizon sub-horizon approx
$x << \frac{c}{aH}$ or $ck >> aH$	$ck >> aH$
Fluid limit: mean free path between galaxies << size of the Universe ~ 1 Mpc << 3000 Mpc	2. The time derivative of the potentials are small compared to the spatial derivatives $\Psi \sim \Psi_0 e^{i\omega \ln a}$ $\omega << \frac{ck}{aH}$ or $ck >> aH\omega$ QSA assumed to hold for Hu-Sawicki model (Hojjati et al. 2012; Silvestri et al. 2013)

Scalar perturbations in the *f(R)* model

ΛCDM	*f(R)*
Continuity equation	Continuity equation
$\frac{\partial \delta}{\partial t} + \left(\frac{\mathbf{v}}{a} \cdot \nabla_x\right)\delta = -\frac{(1+\delta)}{a}(\nabla_x \cdot \mathbf{v})$	$\frac{\partial \delta}{\partial t} + \left(\frac{\mathbf{v}}{a} \cdot \nabla_x\right)\delta = -\frac{(1+\delta)}{a}(\nabla_x \cdot \mathbf{v})$
Euler's equation	Euler equation
$\frac{\partial \mathbf{v}}{\partial t} + \left(\frac{\mathbf{v}}{a} \cdot \nabla_x\right)\mathbf{v} + H\mathbf{v} = -\frac{1}{a}\nabla_x \Psi$	$\frac{\partial \mathbf{v}}{\partial t} + \left(\frac{\mathbf{v}}{a} \cdot \nabla_x\right)\mathbf{v} + H\mathbf{v} = -\frac{1}{a}\nabla_x \Psi$
Poisson's equation	Poisson's equation
$\nabla_x^2 \Psi = \frac{3}{2} H^2 a^2 \Omega_m \delta$	$\nabla_x^2 \Phi_+ = \frac{3}{2} H^2 a^2 \Omega_m \delta$
In absence of anisotropic stress	Einstein's equation (space off-diagonal)
$\Psi = \Phi$	$\nabla_x^2 \chi - \frac{a^2}{3c^2 f_{RR}}\chi = -H^2 a^2 \Omega_m \delta\,.$ where $\Phi_+ = \frac{\Phi + \Psi}{2}$ and $\chi = \Phi - \Psi$ $(\chi = c^2 \delta f_R)$ Pogosian & Silvestri 2008

Scalar perturbations in the *f(R)* model

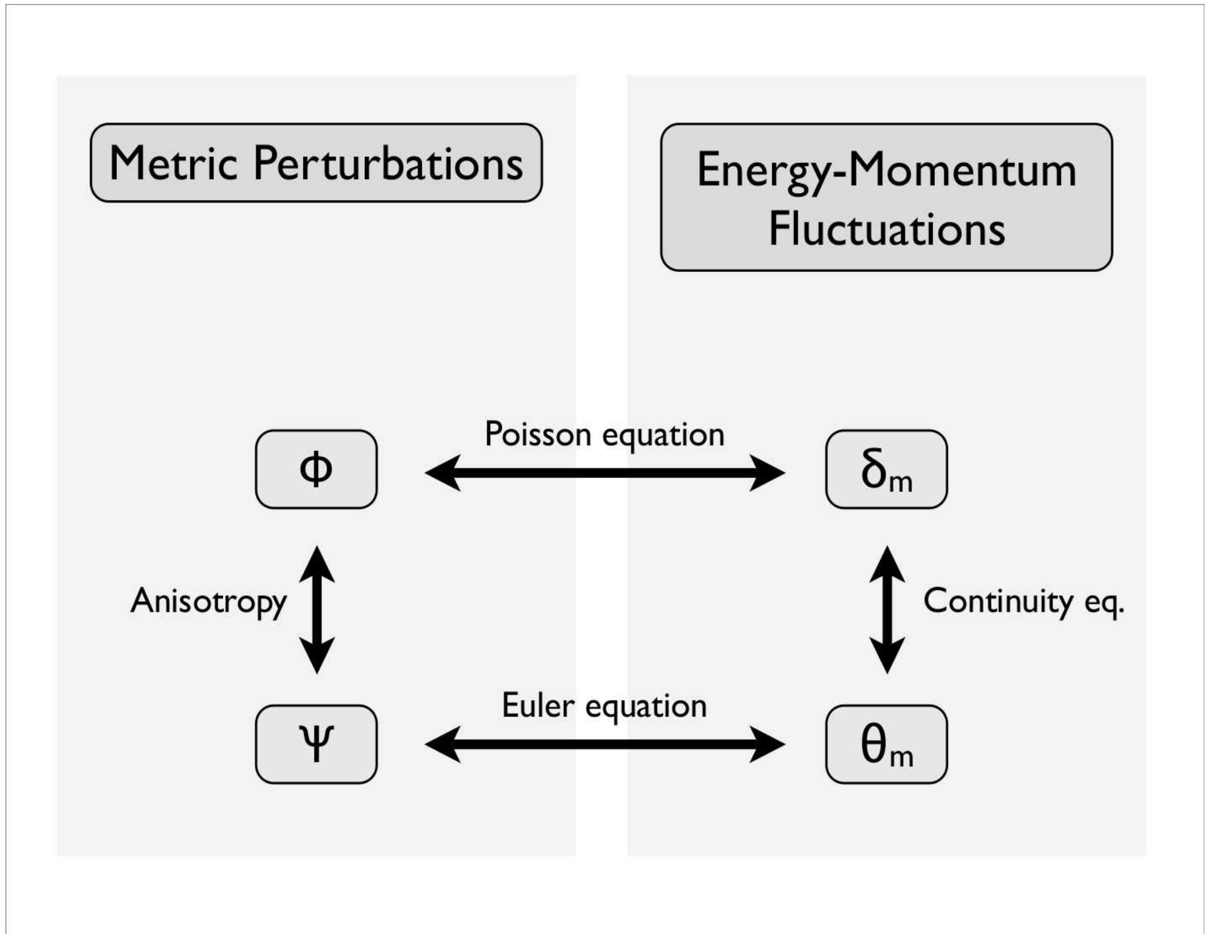


Fig. 1 of Song and Dore JCAP 2009

Strong and Weak regimes

$$\nabla_x^2 \chi - \frac{a^2}{3c^2 f_{RR}} \chi = -H^2 a^2 \Omega_m \delta .$$

f_{RR} is assumed to depend only on the background. Equation for χ is still linear. In Fourier space, it can be written as a modified Poisson equation but not in real space.

$$x_C = \frac{\lambda_C}{a} \quad \text{and} \quad \bar{x}_C = \frac{x_C}{2\pi} \approx \frac{\sqrt{3\tilde{f}_{\tilde{R}\tilde{R}}}}{a}$$

$$\nabla_x^2 \chi - \frac{1}{\bar{x}_C{}^2} \chi = -H^2 a^2 \Omega_m \delta .$$

Define $$Q = \frac{\bar{x}_C}{x_{top}} .$$

- ***Strong Regime (Q >>1):*** *scale of perturbation is much smaller than the Compton wavelength* x_c

$$\nabla_x^2 \chi \approx -H^2 a^2 \Omega_m \delta \qquad \nabla_x^2 \Psi = \left(\frac{4}{3}\right) \frac{3}{2} H^2 a^2 \Omega_m \delta$$ ***Enhanced Newtonian Potential***

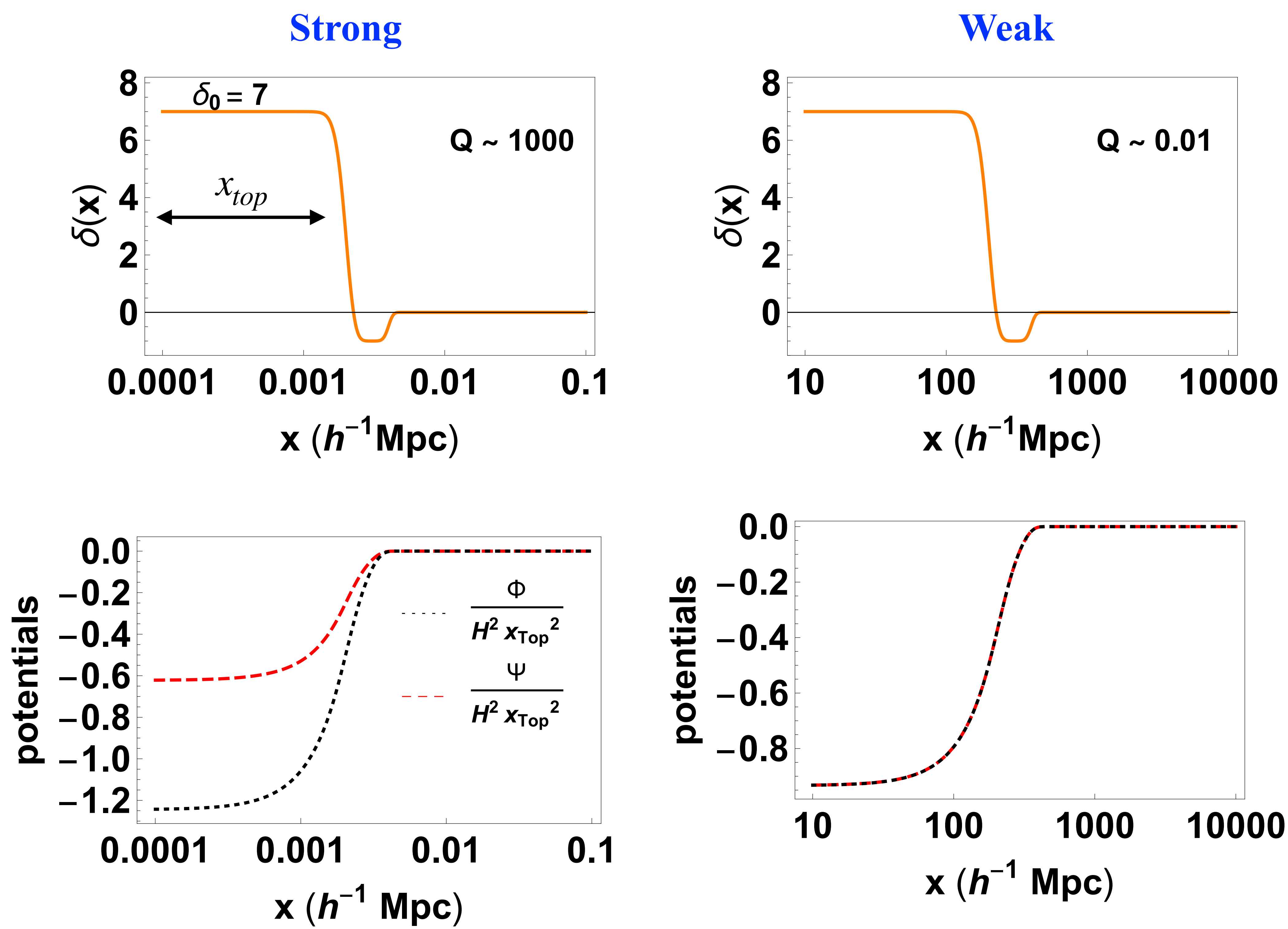
Structure of equations remains unchanged from standard GR

- ***Weak Regime (Q<<1):*** *scale of perturbation is much larger than the Compton wavelength* x_c

$$\chi \sim \bar{x}_C^2 H^2 a^2 \Omega_m \delta \qquad \frac{\chi}{H^2 x_{top}^2} \sim Q^2 H^2 a^2 \Omega_m \delta$$ ***Effect of modification suppressed by*** Q^2

Modification sensitive only to gradient of the density

Potentials in the strong and weak regimes



$f_{R0} = -10^{-6}$ in these plots.

Tools to solve the non-linear regime

N-body simulations

Pros:
- Almost exact

Cons:
- Discrete particle representation Need to Start at relatively low redshifts

Higher order Perturbation Theory

Pros:
- Analytic
- Smooth description
- Non-local

Cons:
- Higher orders cumbersome
- Convergence issues

Restrict to simple geometries

Pros:
- Simple ODEs (in GR)

Cons:
- Non-local

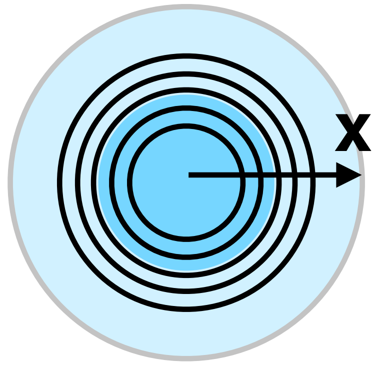
Spherically Symmetric System

For spherically symmetric systems

Physical system: radial density distribution modelled as a series of concentric spherical shells

$$x(q,a) = A(q,a)q \qquad v(q,a) = a\dot{x} = a\dot{A}q = HaA'q$$

q is the initial comoving coordinate of the shell (Lagrangian coordinate)



$\delta(x, a_{init})$

initial velocity profile $$v(x, a_{init}) = -\frac{Ha\Delta(x, a_{init})}{3}$$

ΛCDM	f(R)

ΛCDM

Second order ODE for each shell

$$A'' + \left(2 - \frac{3}{2}\Omega_m(a)\right)A' = -\frac{\Omega_m(a)A}{2}\left(\frac{1+\Delta_{init}}{A^3} - 1\right)$$

Initial conditions

$$A(q, a_{init}) = 1 \qquad A'(q, a_{init}) = \frac{v_{init}(q, a_{init})}{a_{init}H(a_{init})q}$$

$$\Delta(x) = \frac{3}{x^3}\int_0^x \delta(x,a)x^2 dx$$ *spherically averaged density*

$$1 + \Delta(q,a) = \frac{(1+\Delta_{init})A^3_{init}}{A^3}$$

f(R)

Iterative Hybrid Lagrangian-Eulerian scheme

$$A'' + \left(2 - \frac{3}{2}\Omega_m(a)\right)A' = -\frac{1}{q}\nabla_q\tilde{\Psi}(q,a)$$

$$\frac{\partial^2\tilde{\Phi}_+}{\partial x^2} + \frac{2}{x}\frac{\partial\tilde{\Phi}_+}{\partial x} = \frac{3}{2}a^2\Omega_m(a)\delta(x,a)$$

$$\frac{\partial^2\tilde{\chi}}{\partial x^2} + \frac{2}{x}\frac{\partial\tilde{\chi}}{\partial x} - \frac{1}{\bar{x}_C^2}\tilde{\chi} = -a^2\Omega_m(a)\delta(x,a)$$

Boundary conditions

$$\tilde{\Phi}_+(x\to\infty) = 0, \quad \tilde{\chi}(x\to\infty) = 0,$$

$$\left.\frac{\partial\tilde{\Phi}_+}{\partial x}\right|_{x=0} = 0, \quad \text{and} \quad \left.\frac{\partial\tilde{\chi}}{\partial x}\right|_{x=0} = 0.$$

Method of solution

Iterative Hybrid Scheme

- Assume GR at early epochs and solve the second order ODE until $a_{\text{switch}} = 0.1$

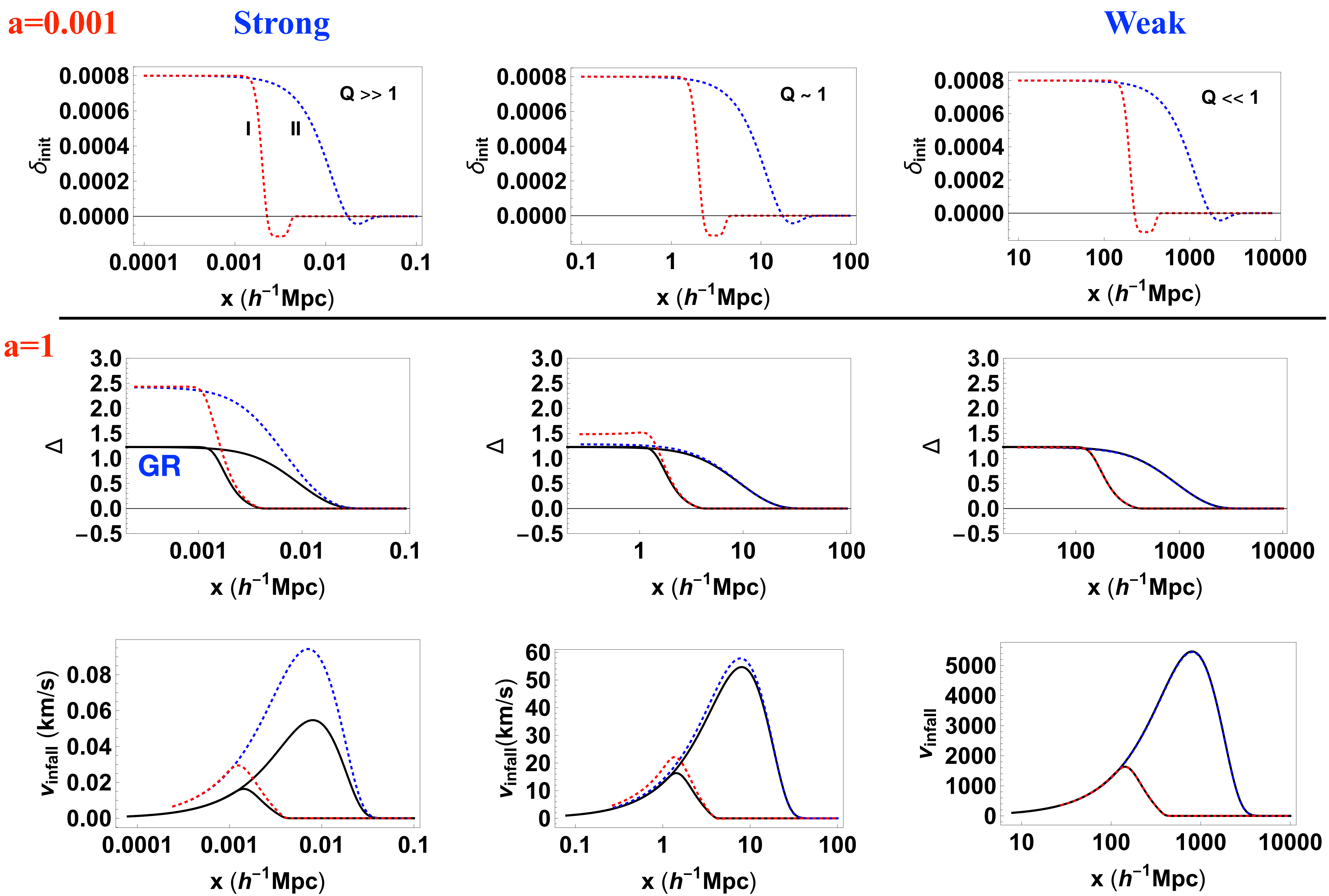
- Divide remaining interval into N_t steps. At each step

First: Solve the spatial equations in the Eulerian domain to get the potentials (analytic solutions available).

Second: Solve the temporal equations for $A(q,a)$ to get the density and velocity at the start of the next step.

Check for convergence as N_t increases.

Non-linearly evolved density and velocity fields



Density-velocity dynamics

Linear Density Velocity Divergence relation

When fluctuations are small, the evolution equations can be linearized

$$\delta'' + \left(2 + \frac{H'}{H}\right)\delta' - \frac{3}{2}\Omega_m(a)\delta = 0.$$

With initial conditions $\delta(a_i), \delta'(a_i)$

General Solution

$$\delta(a) = c_+\delta_+(a) + c_-\delta_-(a)$$

For a pure matter cosmology, $\Omega_m = 1$ analytic solutions

$\delta_+(a) = a$	$\delta_-(a) = a^{-3/2}$
Growing mode	Decaying mode

Impose: ***No perturbations at the big bang i.e set*** $c_- = 0$

Initial conditions no longer independent - only one degree of freedom left.
Density and velocity perturbations are coupled.

Linear Density-Velocity Divergence relation: Linear DVDR
Linearized Continuity Equation

Also the essence of the Zeldovich approximation (Zeldovich 1970)

For a general cosmology

$$\Theta = -f(\Omega_m)\delta = -\Omega_m^\gamma\delta$$

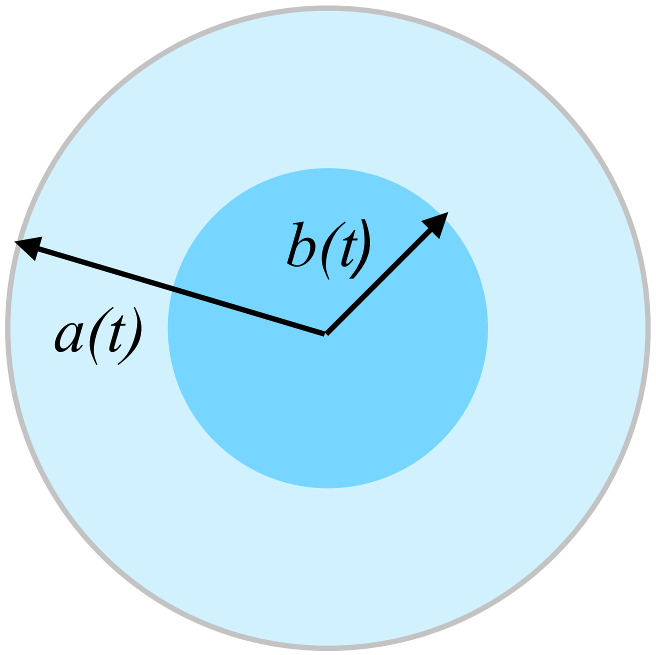
definition $f = \frac{d\ln a}{d\ln\delta}$

Follows from setting no perturbations at big Bang

Growth rate sensitive to cosmological model

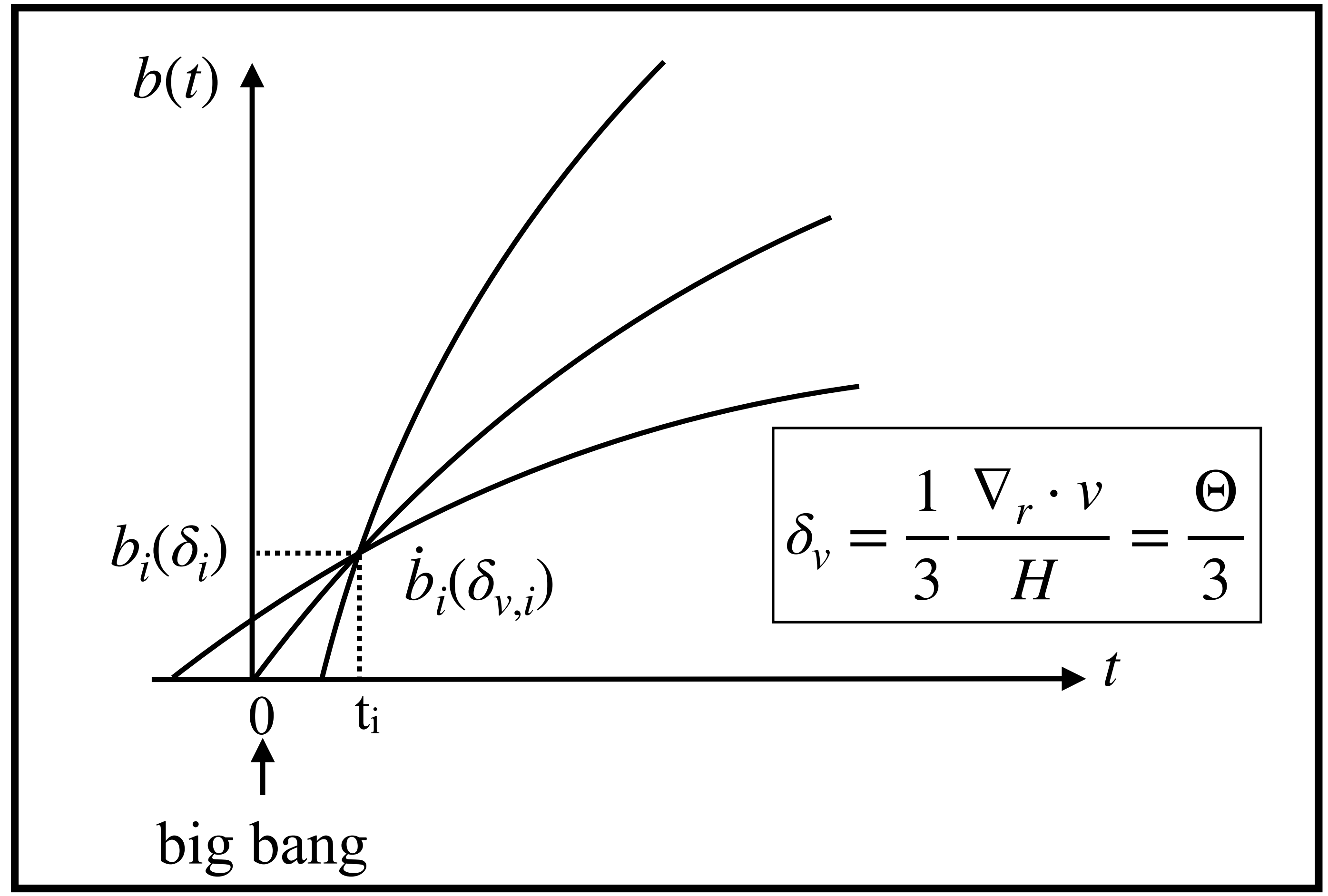
Many observational efforts to constrain it.
Guzzo et al Nature 2008
Majerotto et al 2012
Alam et al 2021
Amendola et al. 2018

Non-linear extension in spherical symmetry



$$\delta = \frac{\rho}{\bar{\rho}} - 1$$

$$\delta_v = \frac{1}{H}\frac{\dot{b}}{b} - 1$$



$$b(t) = a(t)A(t)$$

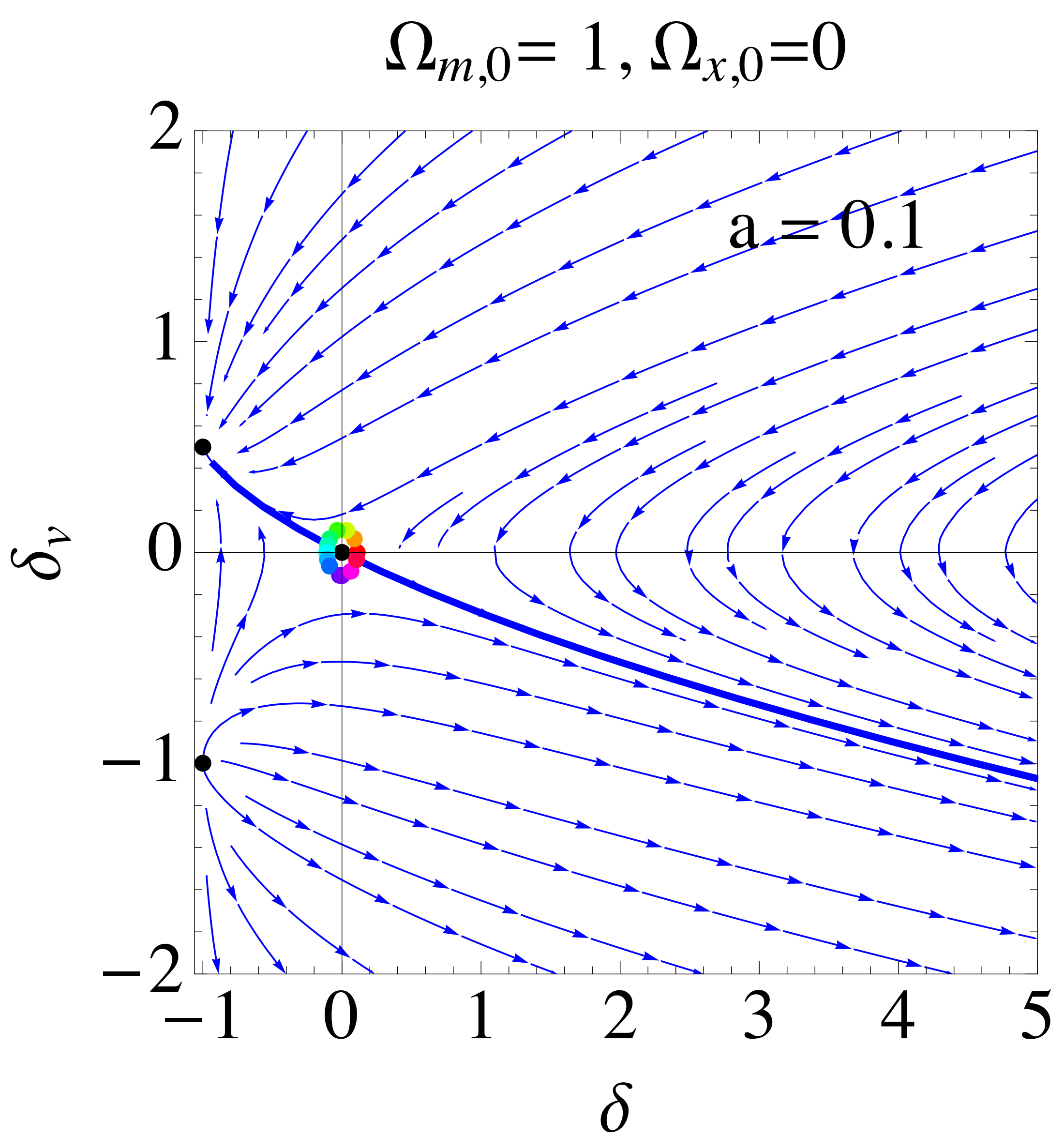
$$A'' + \left(2 - \frac{H'}{H}\right)A' = -\frac{\Omega_m(a)}{2}\left(\frac{1+\delta_i}{A^3} - 1\right)$$

$$A(a_i) = 1,\ A'(a_i) = -\delta_{v,i}$$

For each choice of δ_i choose δ_{vi} such that the solution satisfies no perturbations at the big-bang. Obtain such $\{\delta, \delta_v\}$ pairs for any epoch.

S. N-G **MNRAS** 428 (2013)

Evolution in phase space

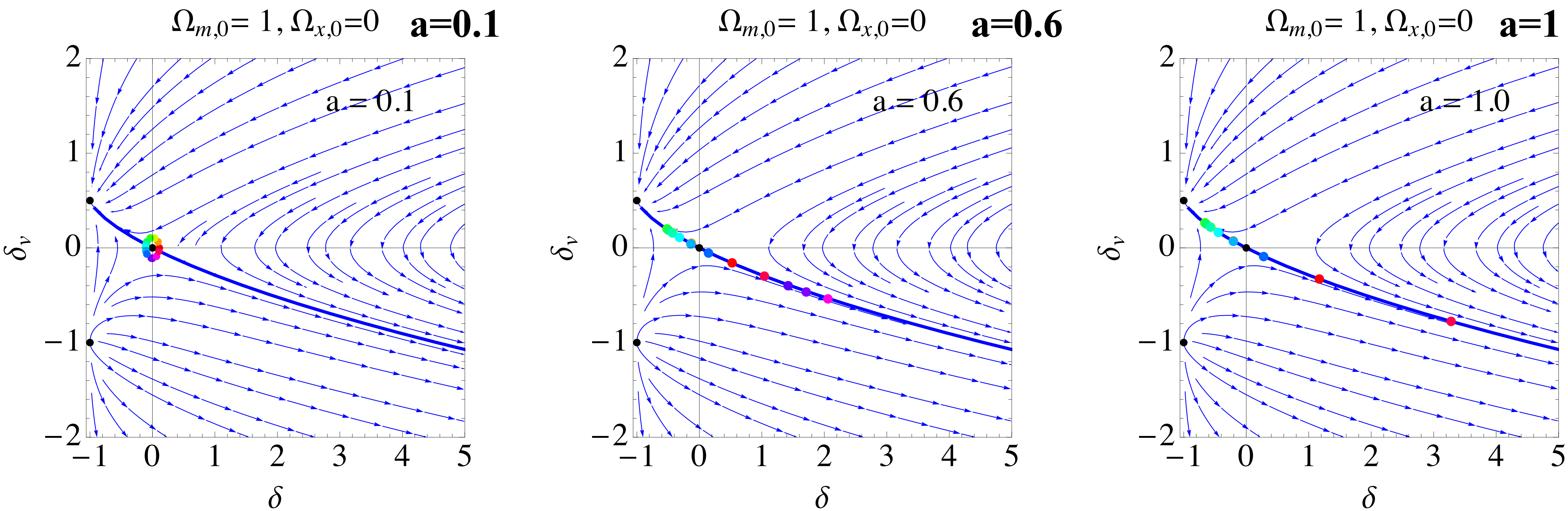


These pairs trace out a very specific curve in a two dimensional δ-δ_v phase-space - `Zeldovich curve'.
No guarantee that this is the late-time non-linear DVDR relation.

Continuity $$\delta' = -3\delta_v(1+\delta)$$

Euler $$\delta_v' = -\frac{1}{2}\left[\Omega_m(a)\delta - \delta_v\left\{\Omega_m(a) + (1+3w)\Omega_\Phi(a) + 2\delta_v^2\right\}\right]$$

DVDR as an invariant set in phase space



Continuity

$$\delta' = -3\delta_v(1+\delta)$$

Euler

$$\delta_v' = -\frac{1}{2}\left[\delta + \delta_v + 2\delta_v^2\right]$$

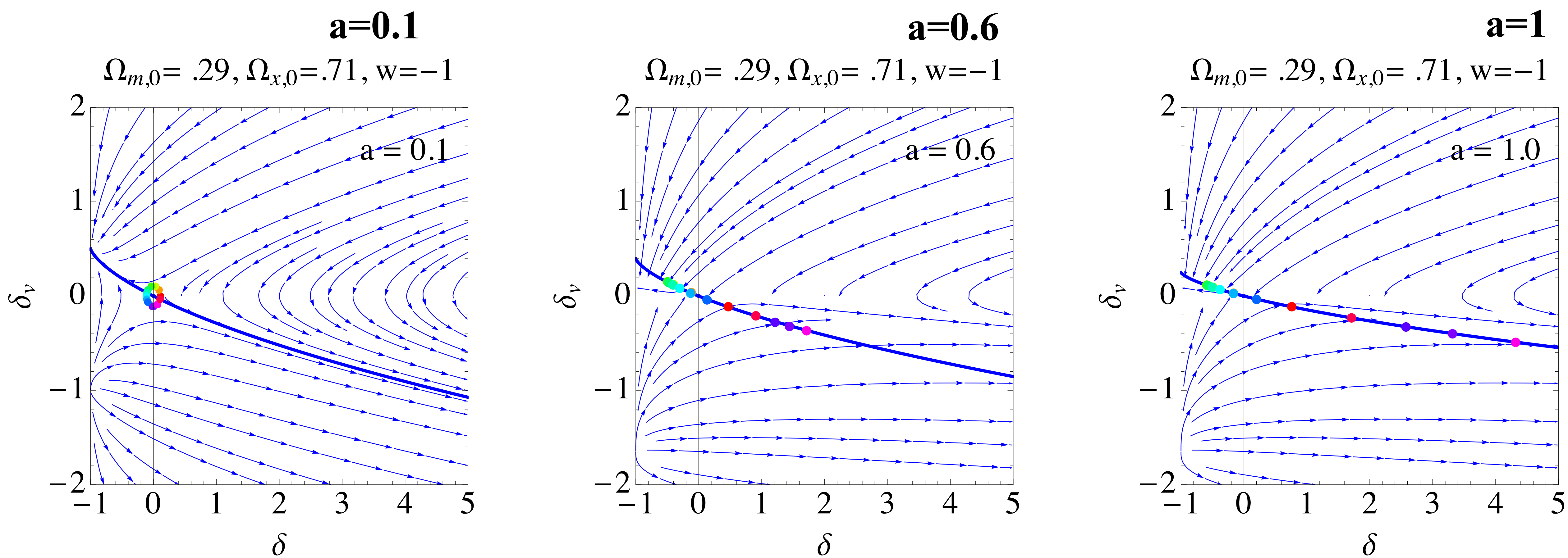
Autonomous dynamical system for

$\Omega_m(a) = 1$

The non-linear DVDR is an invariant set in phase space.

S. N-G. **MNRAS** 428 (2013)

DVDR as an invariant set in phase space



Continuity	$\delta' = -3\delta_v(1+\delta)$
Euler	$\delta_v' = -\frac{1}{2}\left[\Omega_m(a)\delta - \delta_v\left\{\Omega_m(a) + (1+3w)\Omega_\Phi(a) + 2\delta_v^2\right\}\right]$

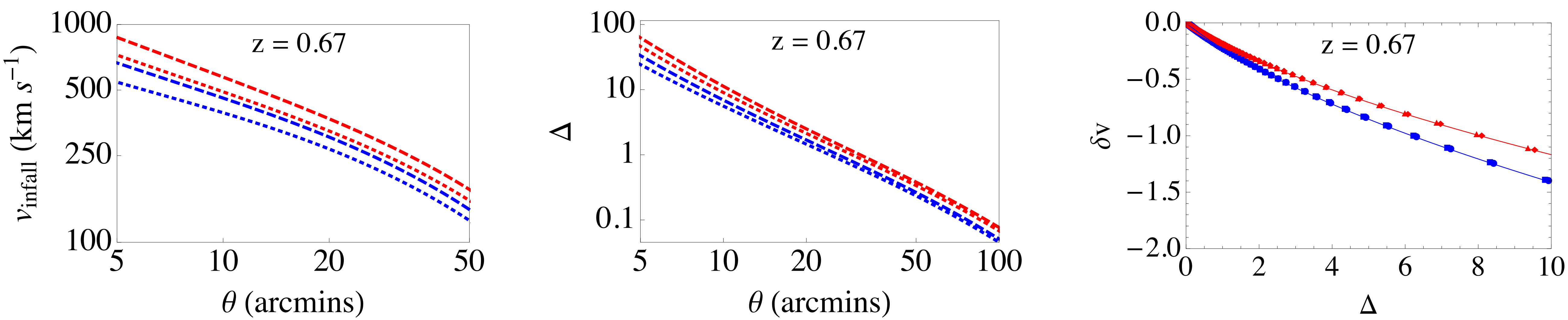
Non-autonomous dynamical system for dark energy models

The non-linear DVDR is an invariant set in phase space.

S. N-G. **MNRAS** 428 (2013)

Removal of parameter degeneracies

Initial profile is radially varying - progenitor profile for a cluster



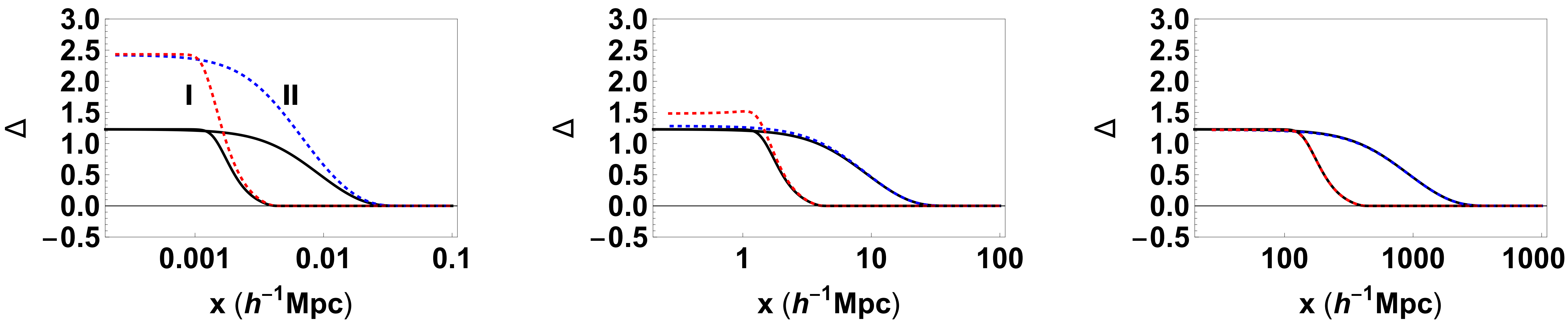
- Red and Blue: *two different values of the dark energy parameter w. Dark energy sets in earlier in the blue so density grows slower.*
- **Dashed** and **Dotted:** *two different values of* σ_8 (denotes the amplitude of initial fluctuations) Dotted is lower σ_8 *so density grows slower.*
- The property of invariant sets can be used exploited to remove parameter degeneracies between these two sets e.g. degeneracy between σ_8 *and w* or σ_8 *and* Ω_m

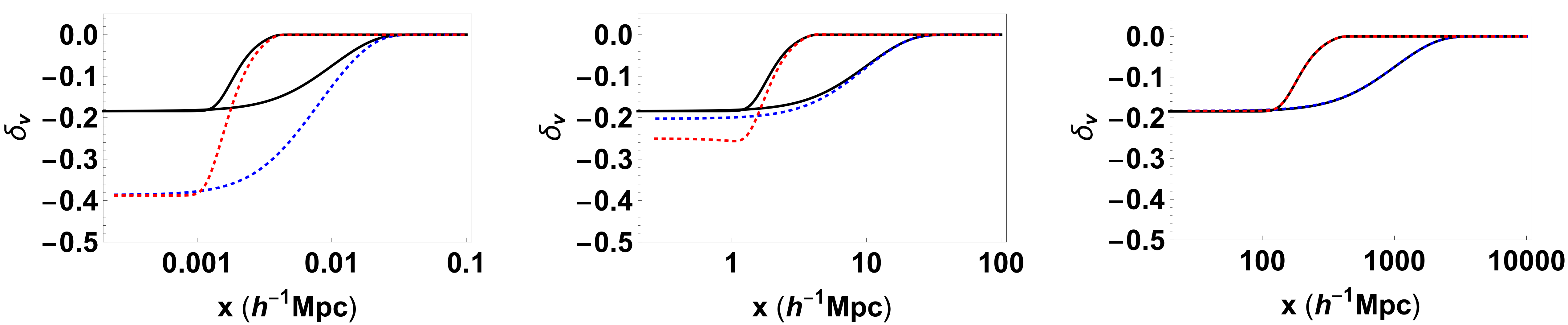
S. N-G. **MNRAS** 428 (2013)

Back to *f(R)*: non-linear DVDR

a=1

Strong **Weak**



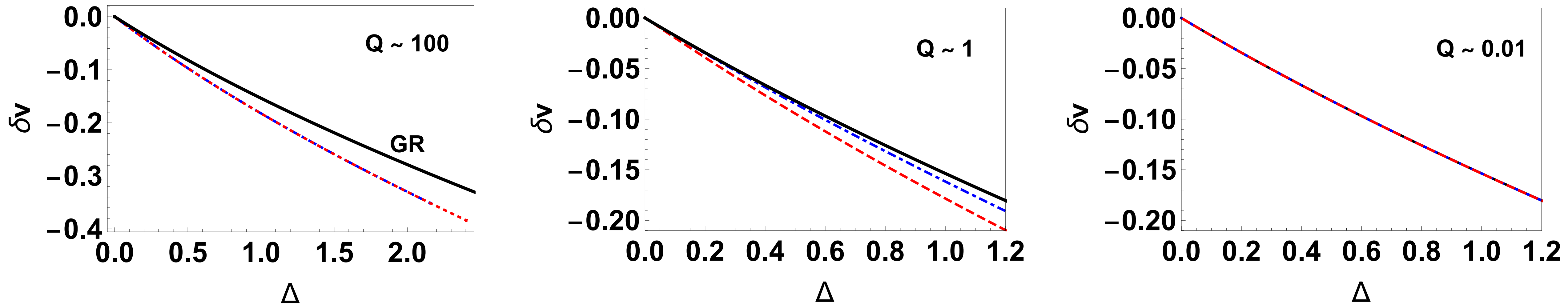


$$\delta_v = \frac{A'}{A}$$

Compute the $\{\Delta, \delta_v\}$ pairs

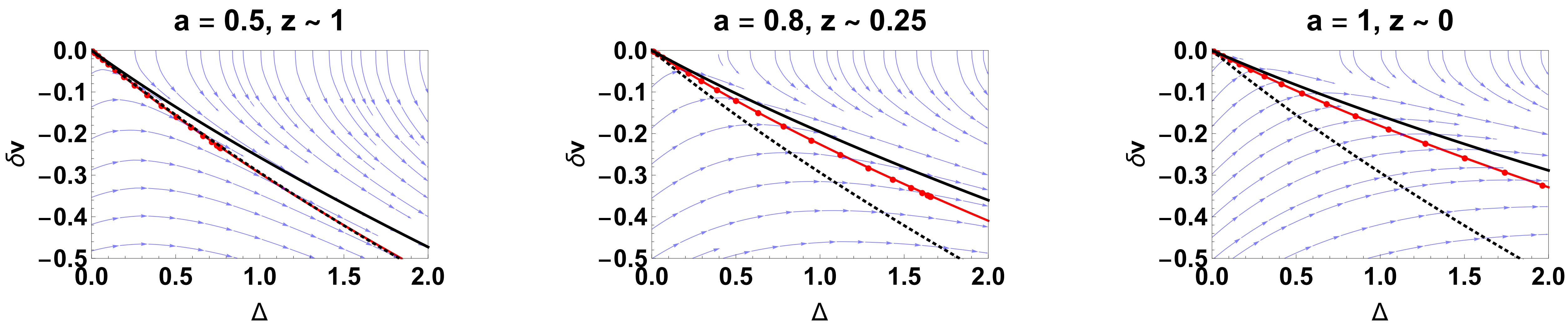
Back to *f(R)*: non-linear DVDR

a=1



- Closed form relation exists only in the strong regime.
- In the intermediate regime, the dynamics is scale dependent and the results are sensitive to the smoothing parameter.
- In the weak field regime, the dynamics seemingly follows GR.
- Does not depend on the switching epoch.

Strong field: phasespace evolution



ΛCDM | ***f(R)***

$$\Delta' = -3(1+\Delta)\delta_v$$

$$\delta_v' = -\frac{1}{2}\Omega_m(a)\Delta - \delta_v\left(2+\frac{H'}{H}\right) - \delta_v^2$$

$$\Delta'_{MG} = -3(1+\Delta_{MG})\delta_{v,MG},$$

$$\delta'_{v,MG} = -\frac{\mathbf{4}}{\mathbf{3}} \times \frac{1}{2}\Omega_m(a)\Delta_{MG} - \delta_{v,MG}\left(2+\frac{H'}{H}\right) - \delta^2_{v,MG}$$

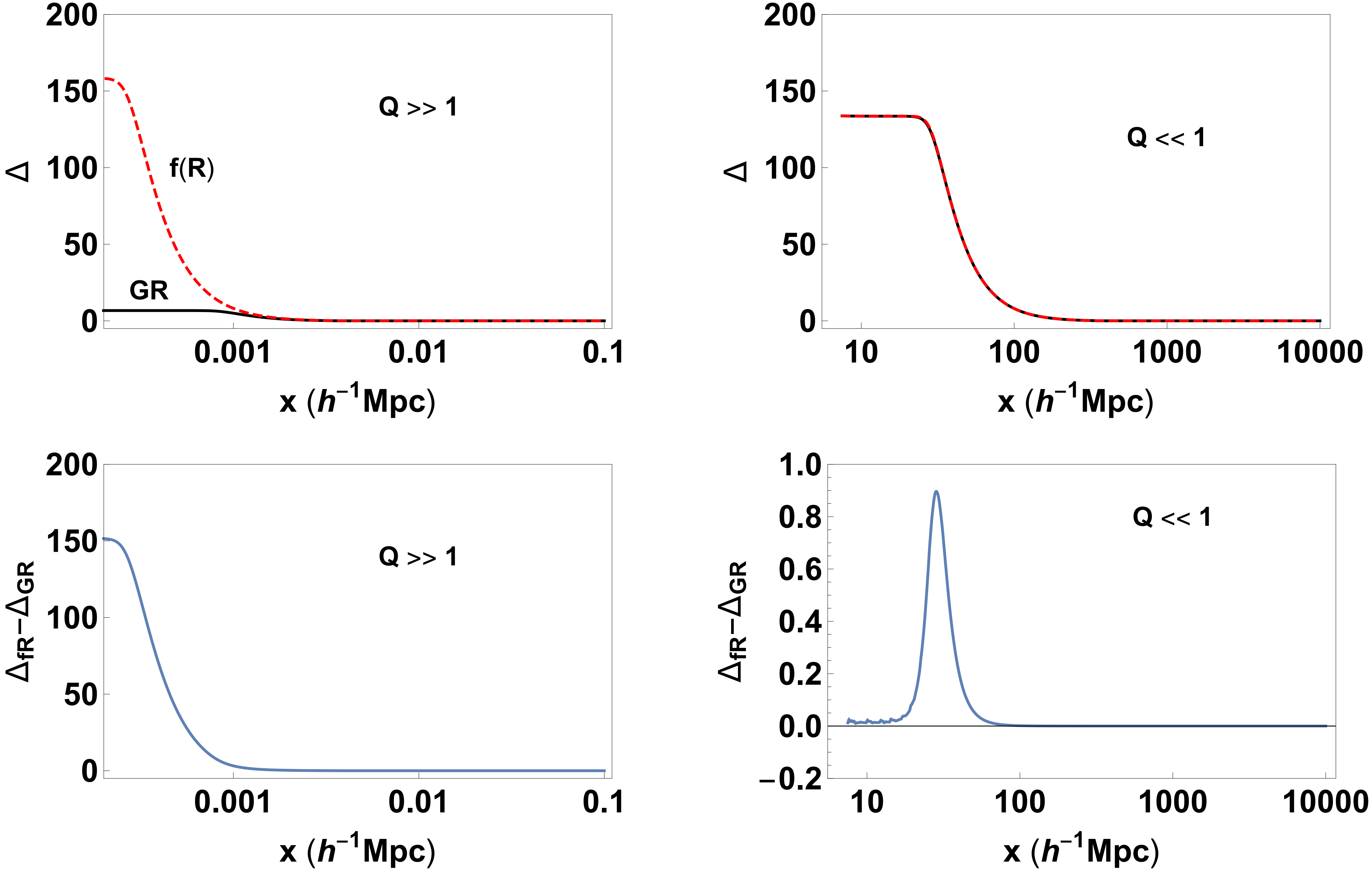
- Red line: fit for *f(R)*.
- Red dots: Points evolved in phase space
- Black solid: ΛCDM
- Black dotted: EdS

Fitting form

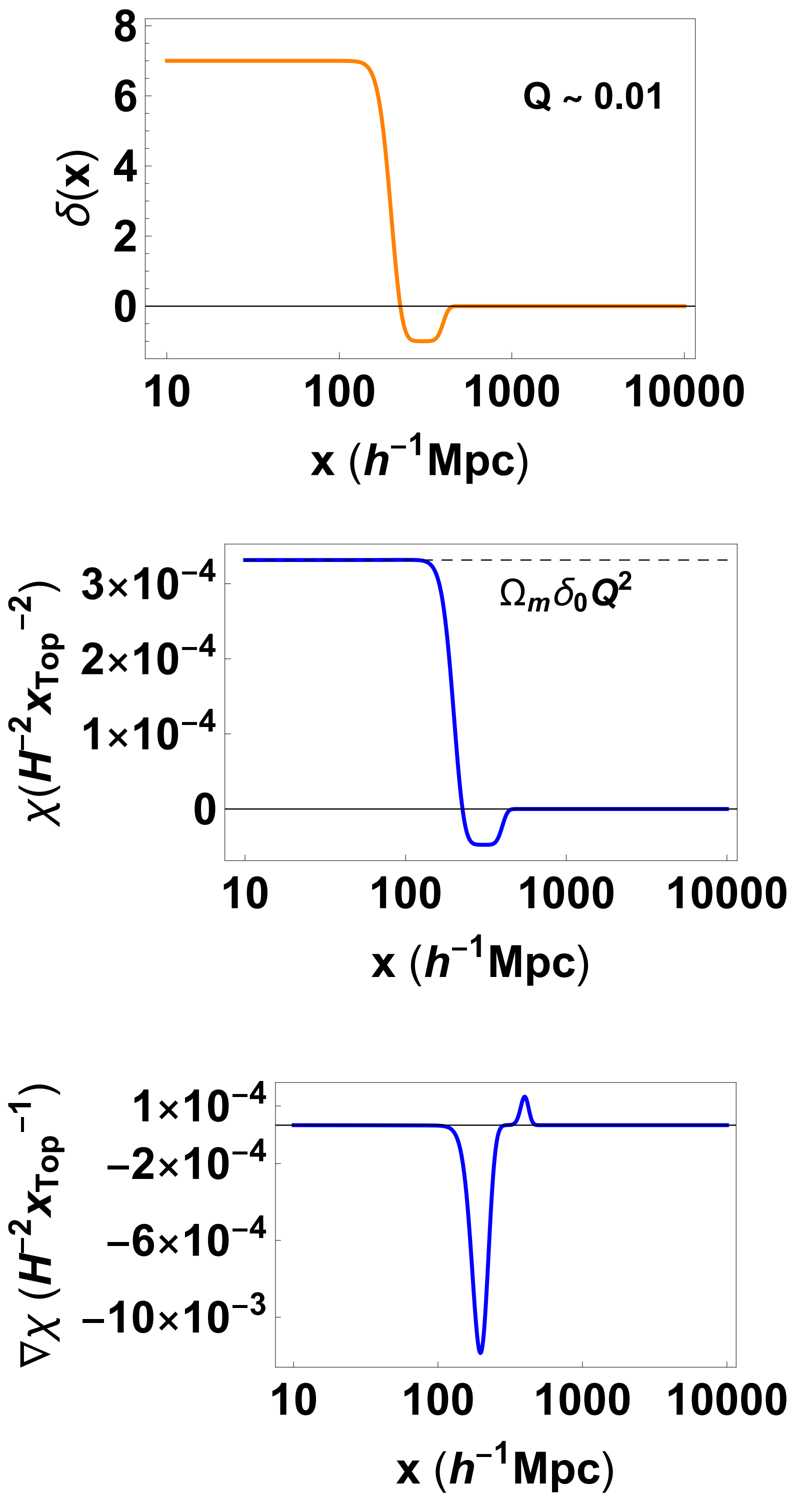
$$\delta_v = 0.64 \times \Omega_m^{0.54}[1-(1+\Delta)^{0.61}]$$

Triaxial collapse: S. N-G and A. Singhal **MNRAS** 457 (2016)
Early Dark Energy models **MNRAS** 498 (2020)

Strongly non-linear evolution



Density enhancement at the edge



- In standard GR, a top-hat remains a top-hat until the innermost shell collapses
- In the *f(R)* theory, a top-hat shows density enhancement at the edge.
- Physically, this is because the edge shells experience a greater force but the inner shells experience the usual acceleration due to GR. This results in a mass accumulation at the edge.
- This has been reported in the literature but in the presence of chameleon screening, where the Compton wavelength decreases inside the top-hat.
- Force is screened in the weak field regime, but not through the Chameleon mechanism.

Borisov et al. 2012; Kopp et al. 2013; Lombriser 2016

- Quantitative estimates and comparisons need further investigation
- But in this regime observable signatures on astrophysical scales where density gradients are large.

Summary and future directions

Main New Features

- Eigen analysis of the Hu-Sawicki background model - helps understand the oscillatory behaviour.
  Application: Investigations of the Quasi-Static Approximation for this and other models.

- Hybrid Lagrangian-Eulerian evolution using analytic solutions.
  Application: Can be also implemented in 3D using Lagrangian PT.
  Nadkarni-Ghosh and Chernoff MNRAS 410, 431 arXiv: 1005.1217,1211.5777
  Analytic solutions can be potentially useful for code checking e.g. SELCIE
  Briddon et al arXiv:2110.11917
  Triaxial collapse - for mock generators like PINOCCHIO
  P. Monaco and collaborators e.g., arXiv:2111.02240

- Phasespace analysis of the density-velocity fields and non-linear DVDR relation.
  Application: Exploit to remove parameter degeneracies. Need to check with simulations.

Main Caveats and possible extensions

- Spherical approximation
- No modelling of screening mechanism which makes the equation for χ non-linear
- Hu-Sawicki model was a working example. Extensions to more general classes of modified gravity such as the EFT formalism
  (Review by Frusciante and Perenon, Physics Reports 2020 arXiv: 1907.03150)