A fast method for quantifying the shape dependence of the bispectrum

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Cosmological random fields

Large-scale cosmological maps contain random fluctuations (more or less NG)

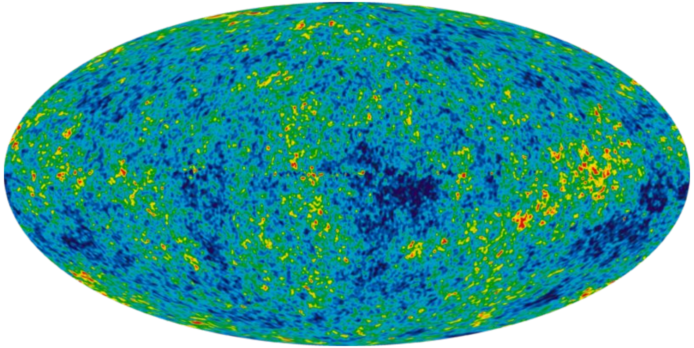
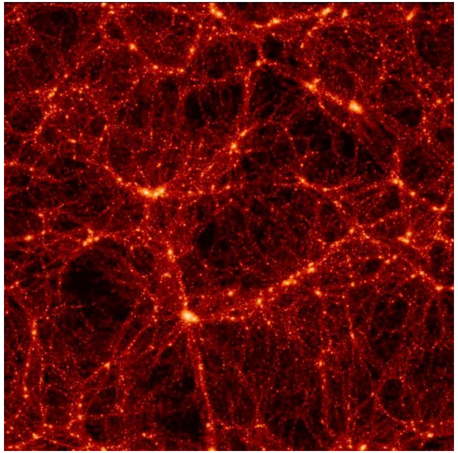
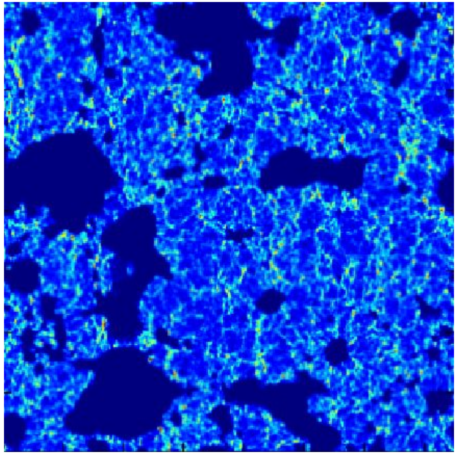


Image credit: WMAP 2010

CMB



DM



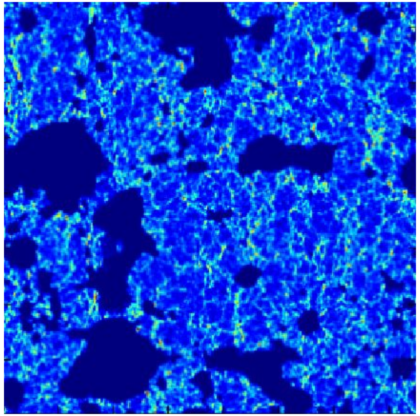
EoR signal

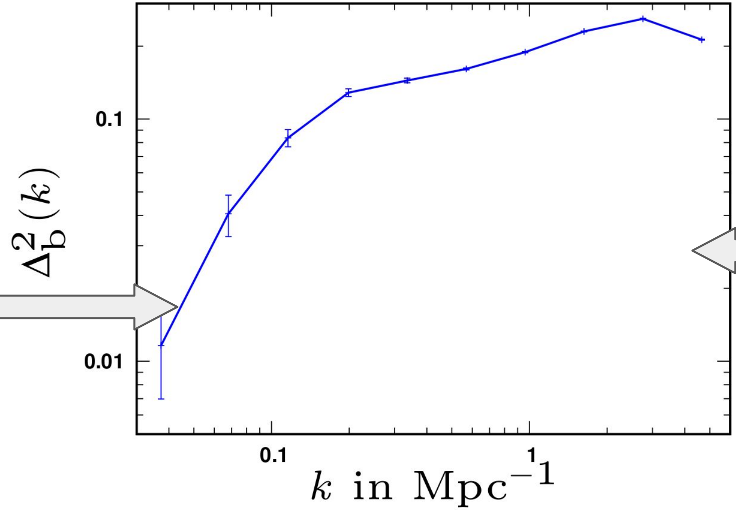
Summary statistics (mostly *power spectrum*) are used to quantify the information encoded in these random fluctuations.

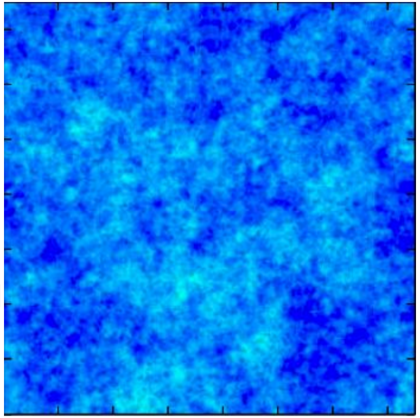
Insufficient Power spectra

Power spectrum is sufficient to quantify fluctuations if the signal is a Gaussian random field.

In order to capture the non-Gaussian features of the field one need higher-order statistics (e.g. bispectrum, trispectrum etc.)

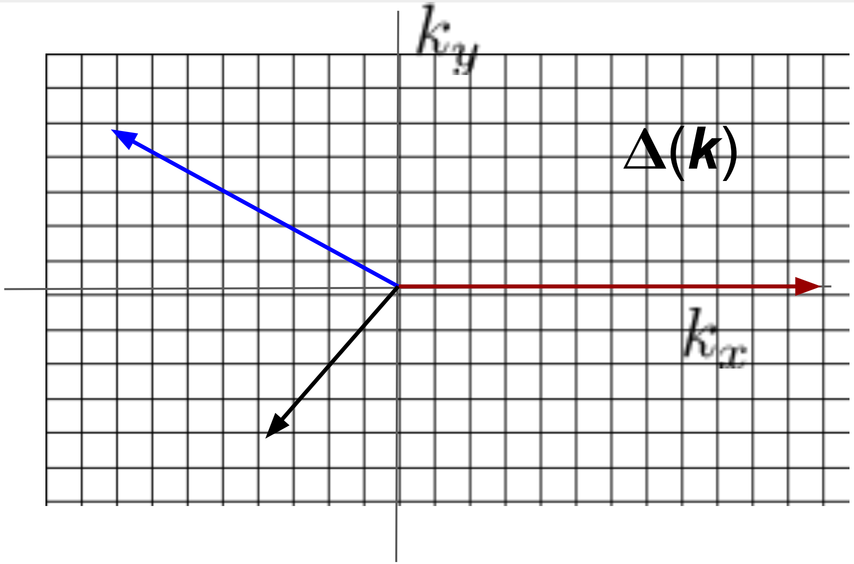






Same power spectrum
But topologically different !

Bispectrum (Three-point statistic)



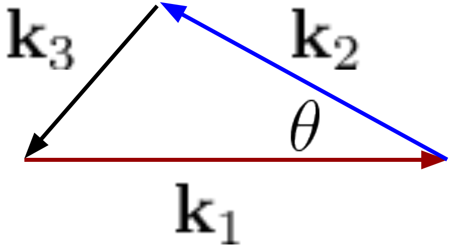
Bispectrum is the higher order statistics next to PS

Sensitive to the non-Gaussianity in the map

Vanishes is the field is Gaussian random

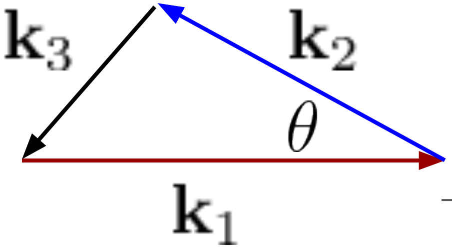
Considering statistical homogeneity

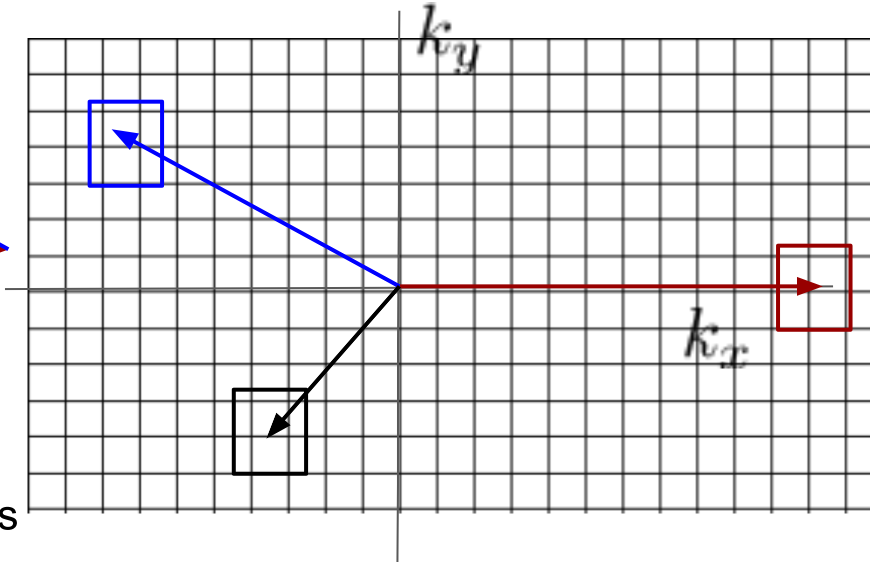
$$\delta_{\rm K}(\vec{k}_1+\vec{k}_2+\vec{k}_3)\ B(k_1,k_2,k_3) = V^{-1}\langle\Delta(\vec{k}_1)\Delta(\vec{k}_2)\Delta(\vec{k}_3)\rangle$$



Binned Bispectrum Estimator

$$\hat{B}(k_1,k_2,k_3) = \frac{1}{V\ N_{\rm tri}} \sum_{\vec{k}_{a_1}} \sum_{\vec{k}_{a_2}} \sum_{\vec{k}_{a_3}} \Delta(\vec{k}_{a_1})\Delta(\vec{k}_{a_2})\Delta(\vec{k}_{a_3})\, \delta_{\rm K}(\vec{k}_{a_1} + \vec{k}_{a_2} + \vec{k}_{a_3})$$





Binning reduces the storage cost
+
decreases the statistical uncertainties in the estimates

Issue

- For a cube with N^3 grids, time required for the direct search approach for closed triangles scales as N^6.
- LSS simulations/ future observations will have large data volume. A direct search method will be extremely time consuming and computationally intractable.
- Need an efficiently faster technique to estimate bispectrum accurately for all possible closed triangles of unique shapes.

Fourier Technique

Numerator

$$\hat{B}(k_1,k_2,k_3) = \frac{1}{V\,N_{\rm tri}} \sum_{\vec{k}_{a_1}} \sum_{\vec{k}_{a_2}} \sum_{\vec{k}_{a_3}} \Delta(\vec{k}_{a_1})\Delta(\vec{k}_{a_2})\Delta(\vec{k}_{a_3})\, \delta_{\rm K}(\vec{k}_{a_1} + \vec{k}_{a_2} + \vec{k}_{a_3})$$

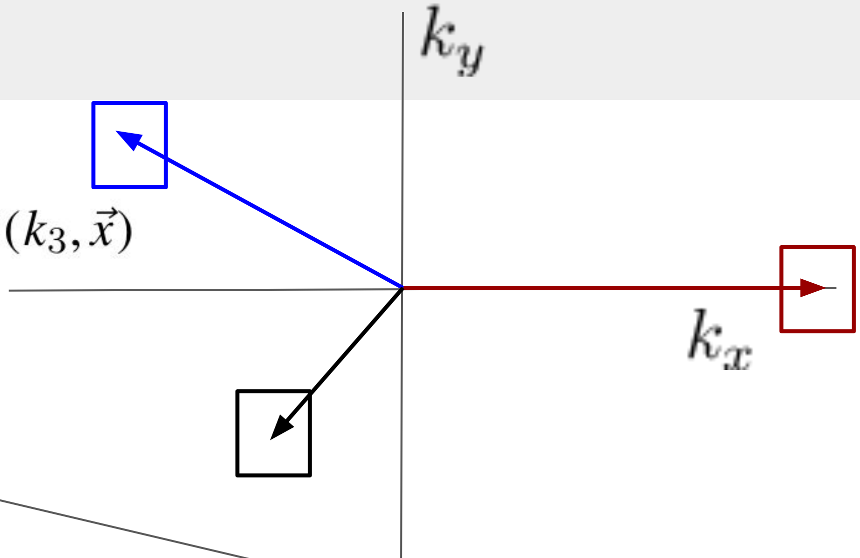
Denominator

$$N_{\rm tri} = \sum_{\vec{k}_{a_1}} \sum_{\vec{k}_{a_2}} \sum_{\vec{k}_{a_3}} \delta_{\rm K}(\vec{k}_{a_1} + \vec{k}_{a_2} + \vec{k}_{a_3})$$

Evaluate the sum using

$$\delta_{\rm K}(\vec{k}_{a_1} + \vec{k}_{a_2} + \vec{k}_{a_3}) = \frac{1}{N_{\rm g}^3} \sum_{\vec{x}} \exp\left(-i[\vec{k}_{a_1} + \vec{k}_{a_2} + \vec{k}_{a_3}] \cdot \vec{x}\right)$$

Fast Fourier Transform



$$\hat{B}(k_1,k_2,k_3) = \frac{1}{V}\frac{1}{N_{\rm tri}}\frac{1}{N_{\rm g}^3}\sum_{\vec{x}} D(k_1,\vec{x})D(k_2,\vec{x})D(k_3,\vec{x})$$

Where,

$$D(k_n,\vec{x}) = \sum_{\vec{k}_{a_n}} \Delta(\vec{k}_{a_n})\exp\left(-i\vec{k}_{a_n}\cdot\vec{x}\right)$$

$$N_{\rm tri} = \frac{1}{N_{\rm g}^3}\sum_{\vec{x}} I(k_1,\vec{x})I(k_2,\vec{x})I(k_3,\vec{x})$$

$$I(k_n,\vec{x}) = \sum_{\vec{k}_{a_n}} \exp\left(-i\vec{k}_{a_n}\cdot\vec{x}\right)$$

Using Fast Fourier Transform

Total **6** FFTs are required.

$O(N^3 \log N^3)$ time-complexity

Bispectrum Parameterization

We design the estimator to compute the bispectrum for the triangles of all possible **shapes and sizes**.

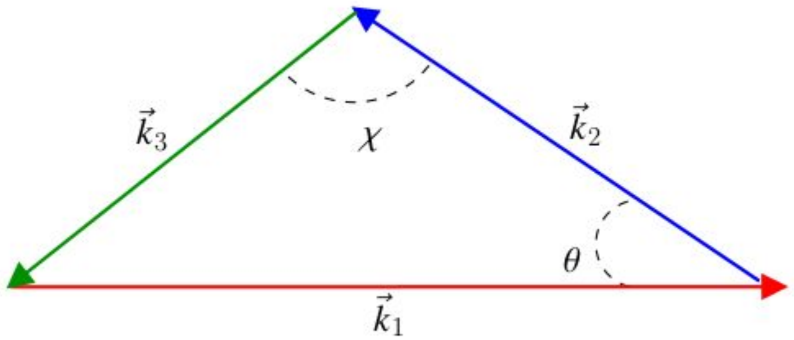
Size : k_1 (length of the largest side)

Shape : $\begin{cases} t = k_2/k_1 \\ \mu = \cos\theta = -(\boldsymbol{k}_1 \cdot \boldsymbol{k}_2)/(k_1 k_2) \end{cases}$

$$B(k_1, k_2, k_3) \leftrightarrow B(k_1, \mu, t)$$

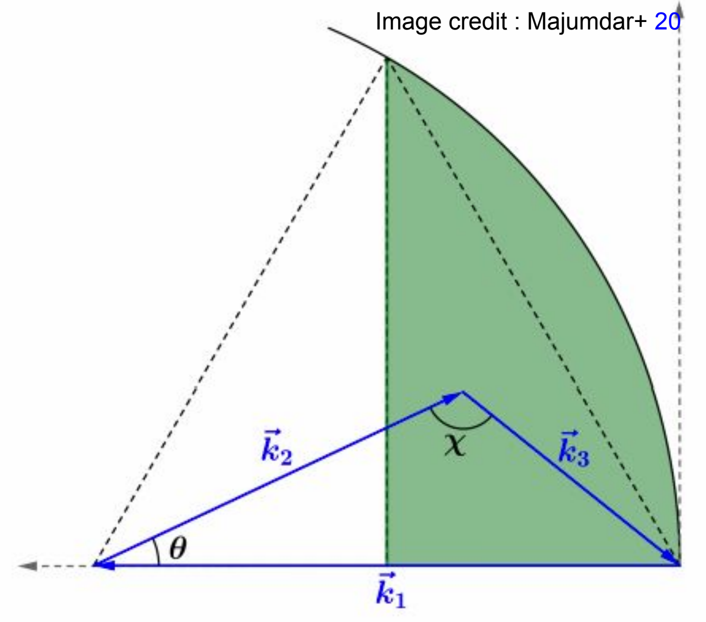
Unique triangles (Bharadwaj+ 20)

$$k_1 \geq k_2 \geq k_3$$



Unique shaped triangles

Image credit : Majumdar+ 20



$$k_1 \geq k_2 \geq k_3$$

$$0.5 \leqslant t, \mu \leqslant 1.0 \quad \text{and} \quad 2t\mu \geqslant 1$$

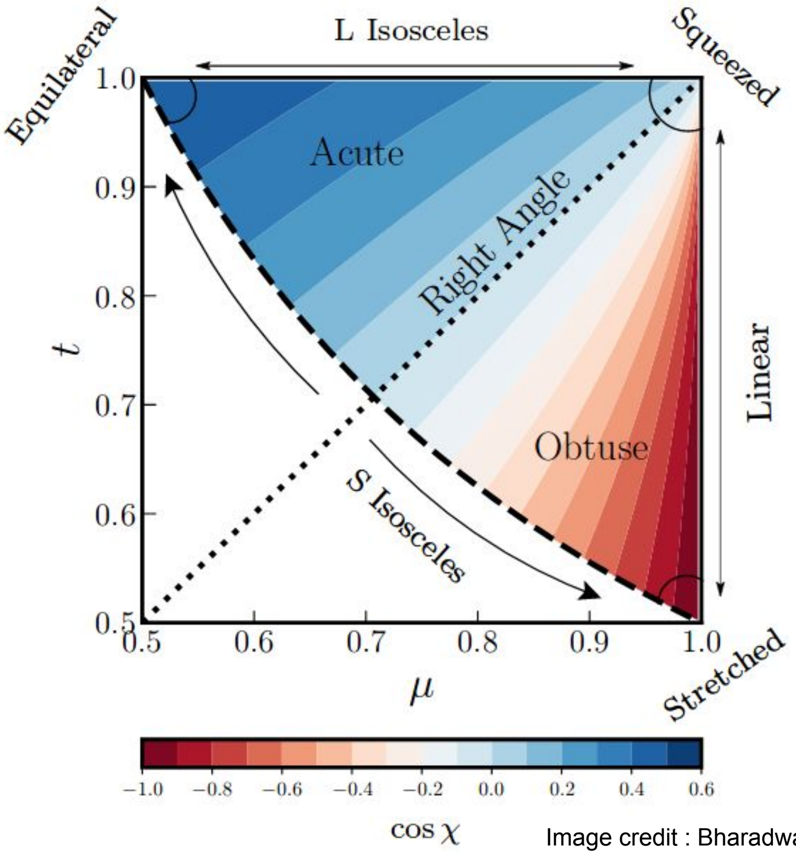
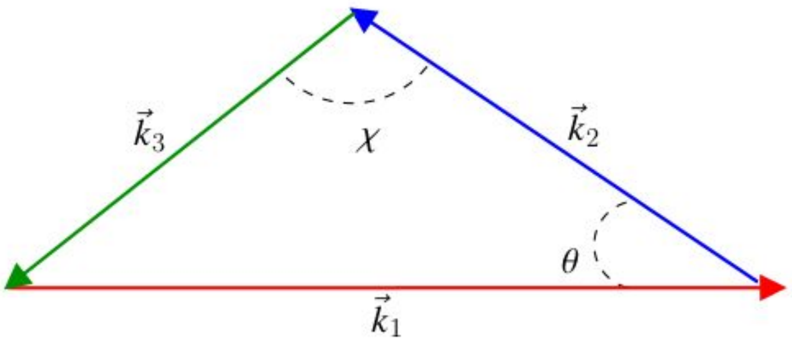


Image credit : Bharadwaj+ 20

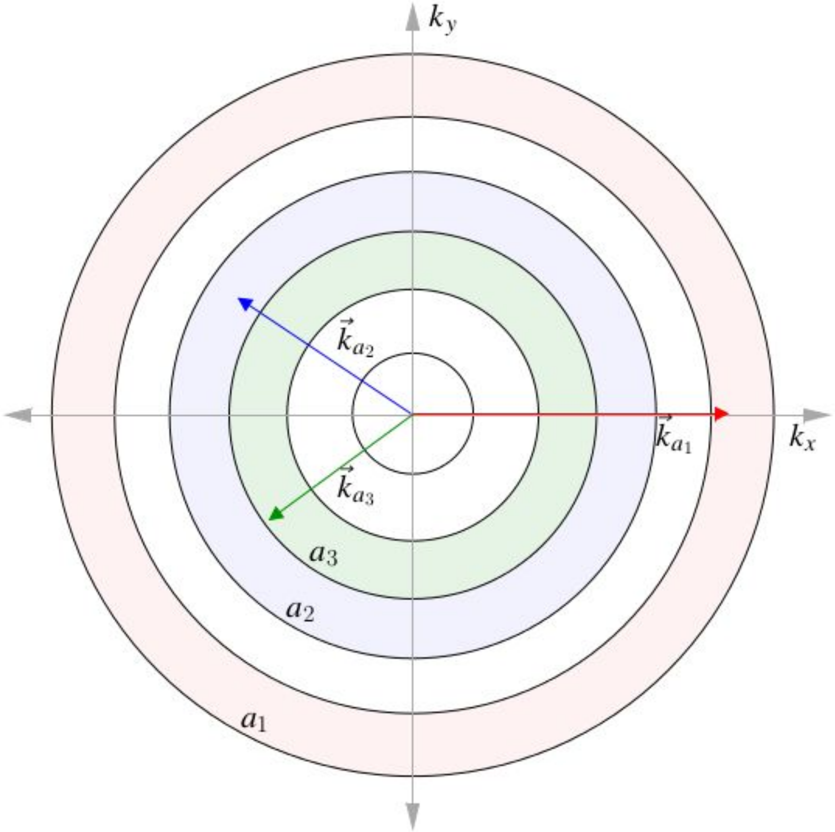
Binning scheme

Average over all possible orientation of a triangle in 3D Fourier space.

use **circular rings/spherical shells** in ***k***-space.



We compute binned bispectra using combinations of three bins which satisfy the unique triangle conditions.



Sampling μ–t plane

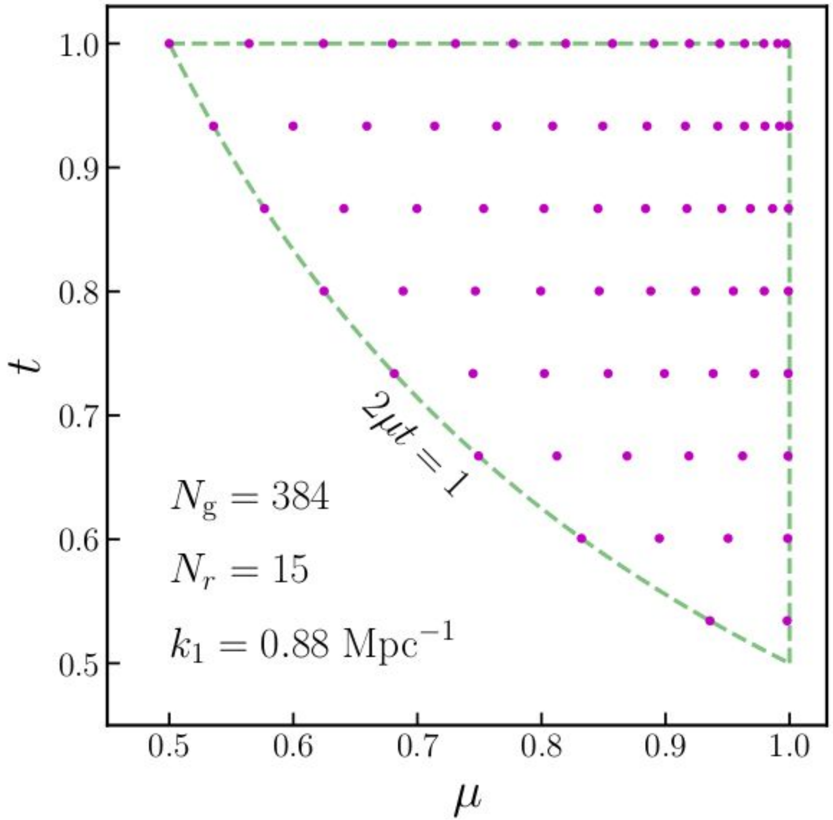
$$t = \frac{k_2}{k_1}$$

$$\mu = \frac{1}{2}\left[\frac{k_1}{k_2} + \frac{k_2}{k_1} - \frac{k_3}{k_1}\frac{k_3}{k_2}\right]$$

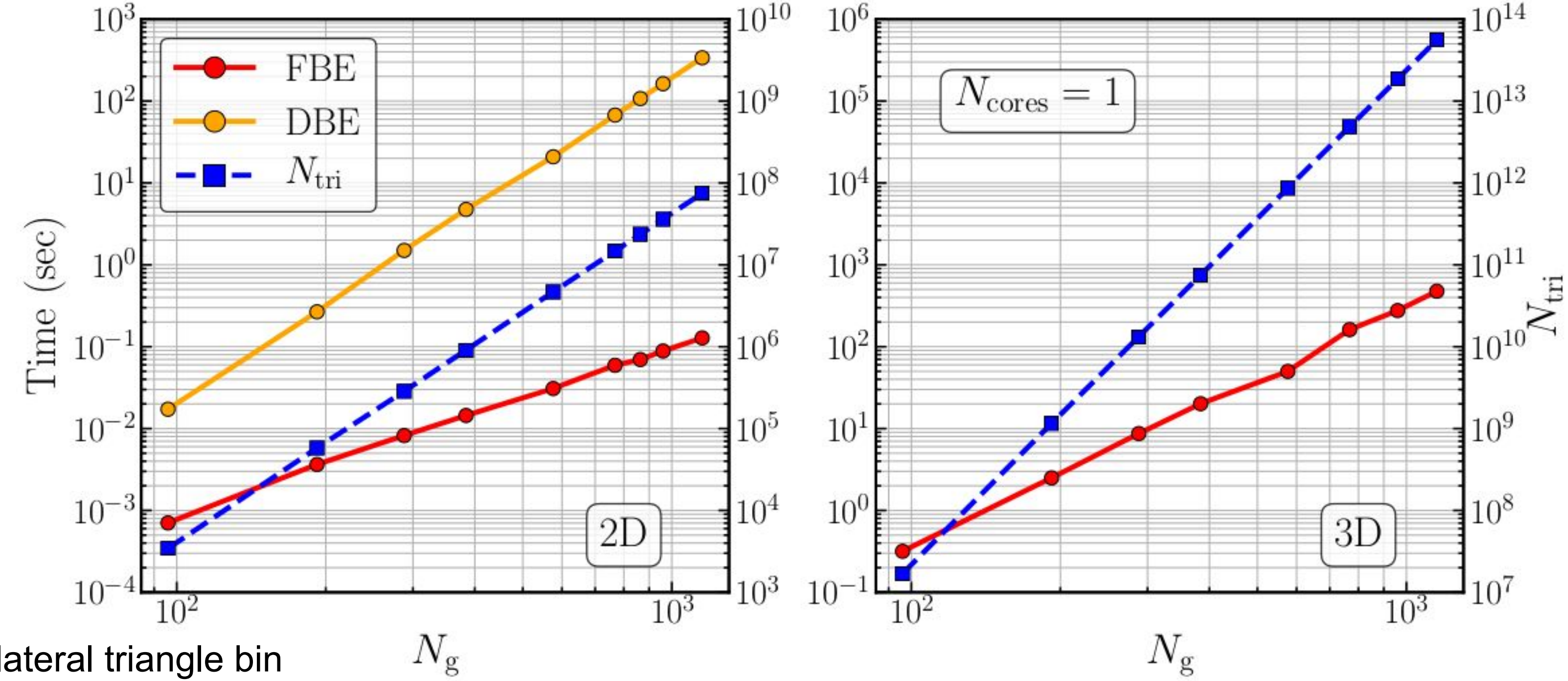
Increasing more number of shells will increase the sampling of μ-t plane.

However smaller shell thickness increase the fluctuations in the estimated binned-bispectra.

The computation time will also increase due to more numbers of shells.



Performance



Equilateral triangle bin

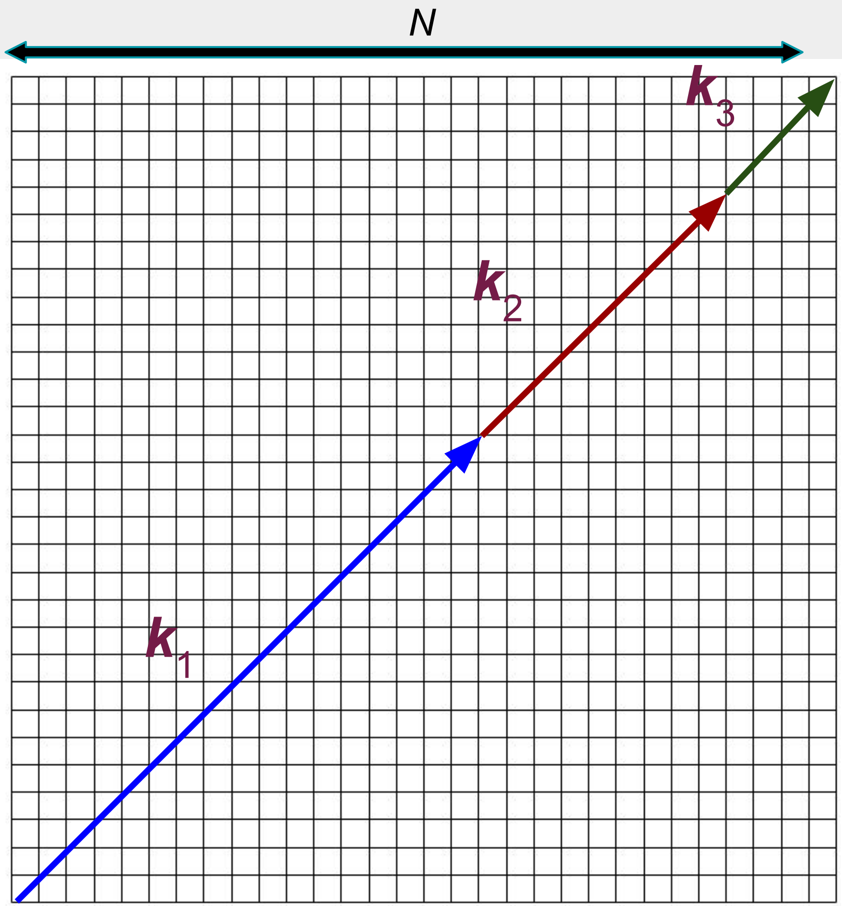
N_g = 100 ---> 0.3 sec (FBE) | 150 sec (DBE)

Highly time efficient !

N_g = 1000 ---> 300 sec (FBE) | 1.5×10^{8} sec (DBE)

Bottleneck beyond N/3

If one of the three vectors crosses N/3 on the Fourier grid there could be several instances where the sum of the three vectors will hit the other boundary of the box



Bottleneck beyond N/3

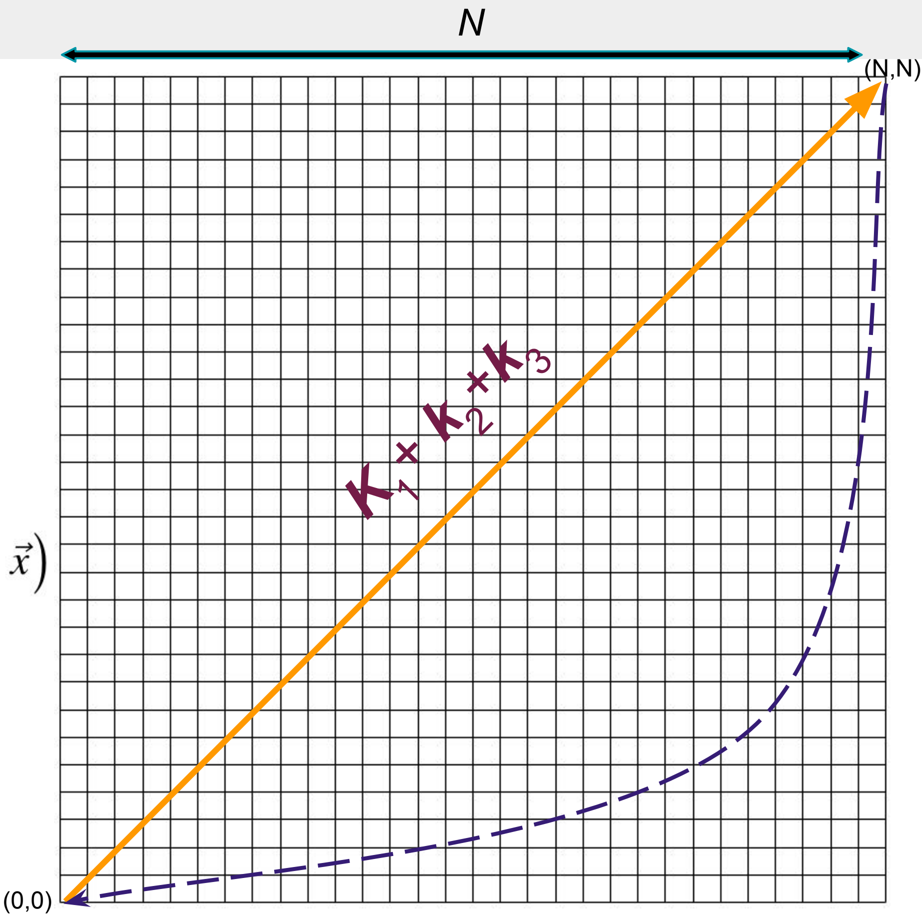
If one of the three vectors crosses N/3 on the Fourier grid there could be several instances where the sum of the three vectors will hit the other boundary of the box

The sum then will wrap around the box and map back to the origin due to periodicity in the Fourier space

$$\delta_{\rm K}(\vec{k}_{a_1}+\vec{k}_{a_2}+\vec{k}_{a_3}) = \frac{1}{N_{\rm g}^3}\sum_{\vec{x}}\exp\left(-i[\vec{k}_{a_1}+\vec{k}_{a_2}+\vec{k}_{a_3}]\cdot\vec{x}\right)$$

The factor in the exponent will become integral multiple of 2π

$$\vec{k}_{a_1}+\vec{k}_{a_2}+\vec{k}_{a_3} = \vec{q}\left(\frac{2\pi N_g}{L}\right)$$



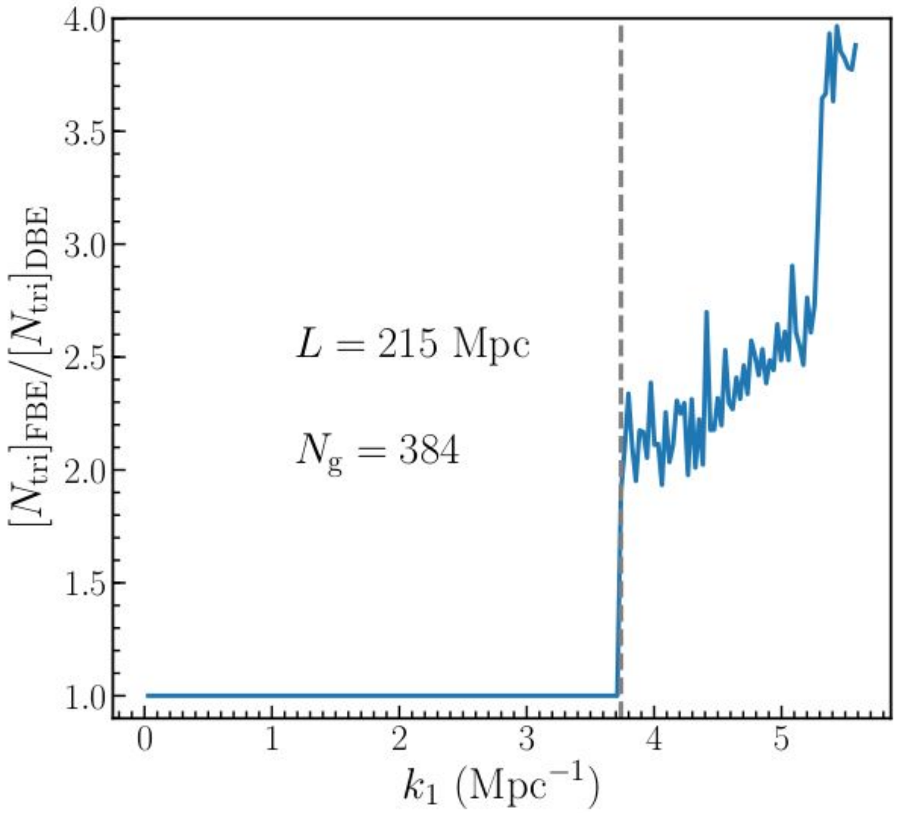
Excess triangles !

The FFT method predicts more number of triangles (due to PBC) than the true number of closed triangles

Counted exact number of triangles in equilateral triangle bin using direct method estimator.

For bispectrum the computation must be restricted within $(2\pi N_g)/(3L)$

For any *p*-th order spectrum *k* < $(2\pi N_g)/(pL)$



Validating FBE

Second-order biased NG field

- Generated Gaussian random field for a given power-law power spectrum $P(k)=Ak^n$.

e.g. A = 1, n = -2

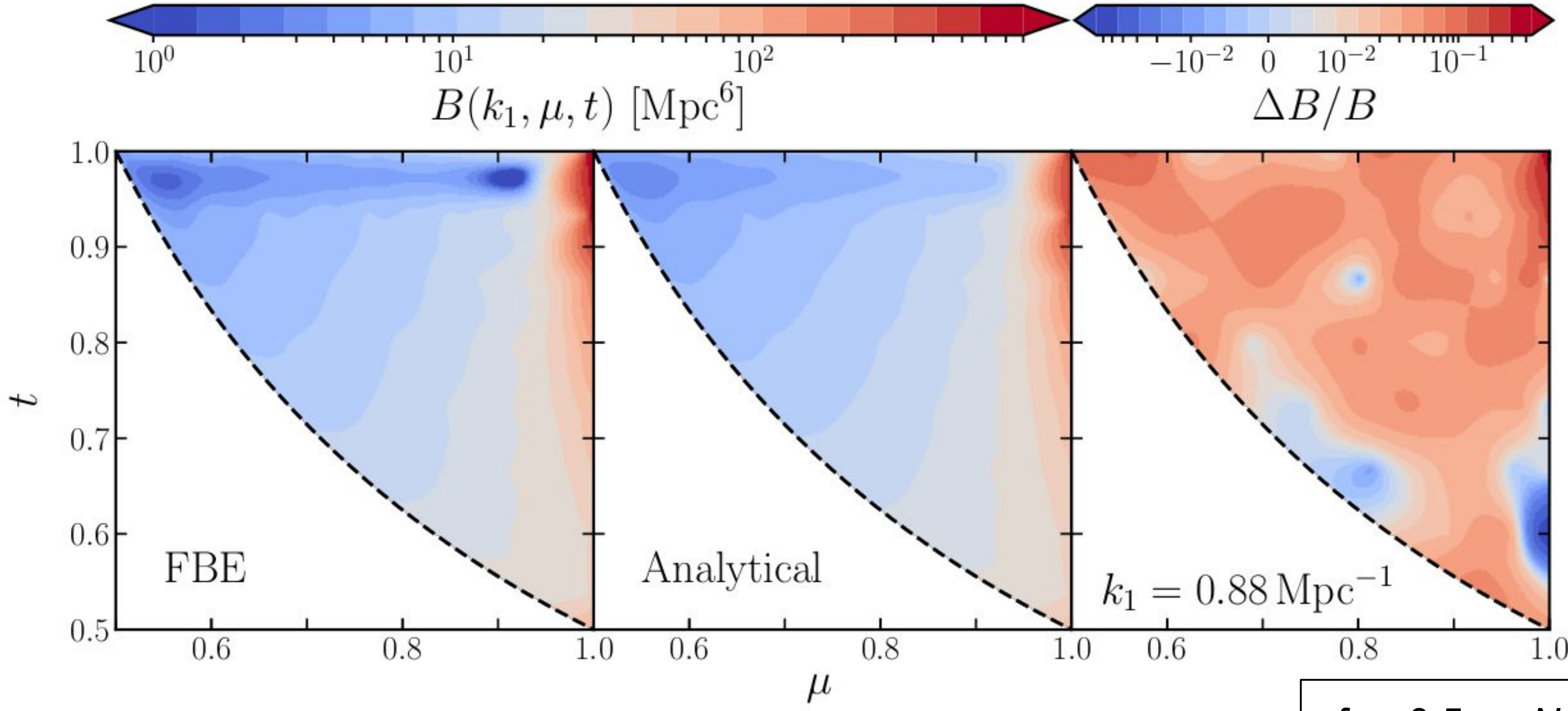
- Introduced non-Gaussianity using :

$$\delta(\vec{x}) = \delta_{\rm G}(\vec{x}) + f_{\rm NG}[\delta^2_{\rm G}(\vec{x}) - \langle\delta^2_{\rm G}(\vec{x})\rangle]$$

- Compared the FBE results against the tree-level analytical predictions (for mild NG)

$$B_{\rm Ana}(k_1, k_2, k_3) = 2f_{\rm NG}[P(k_1)P(k_2) + P(k_2)P(k_3) + P(k_3)P(k_1)]$$

Results-I (Real space)



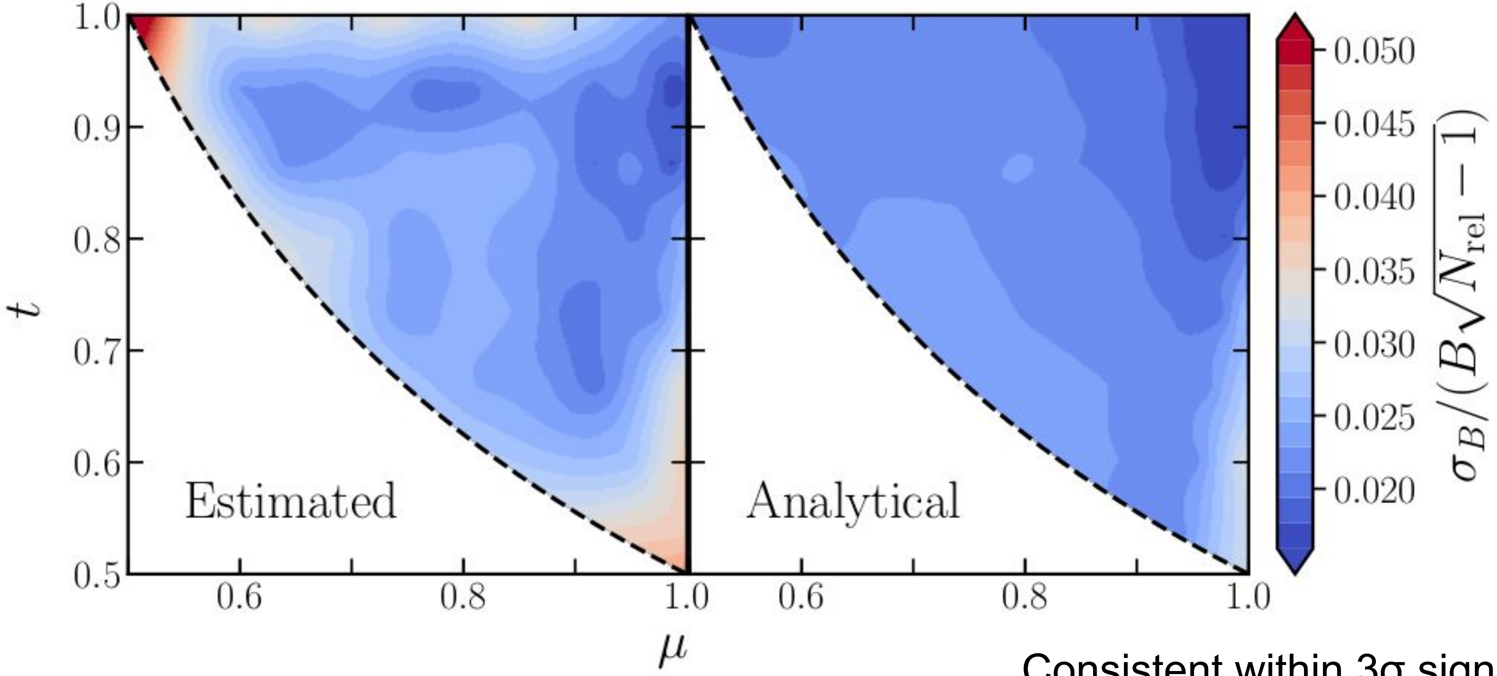
Mean bispectrum computed over an ensemble of **100 realizations**

f_{NG}=0.5; N_g=384

Results-I (Real space)

Compared with Analytical CV

$$\sigma_B^2 = \frac{1}{N_{\rm tri}}[VP(k_1)P(k_2)P(k_3) + 3B^2(k_1,k_2,k_3)]$$



Consistent within 3σ significance.

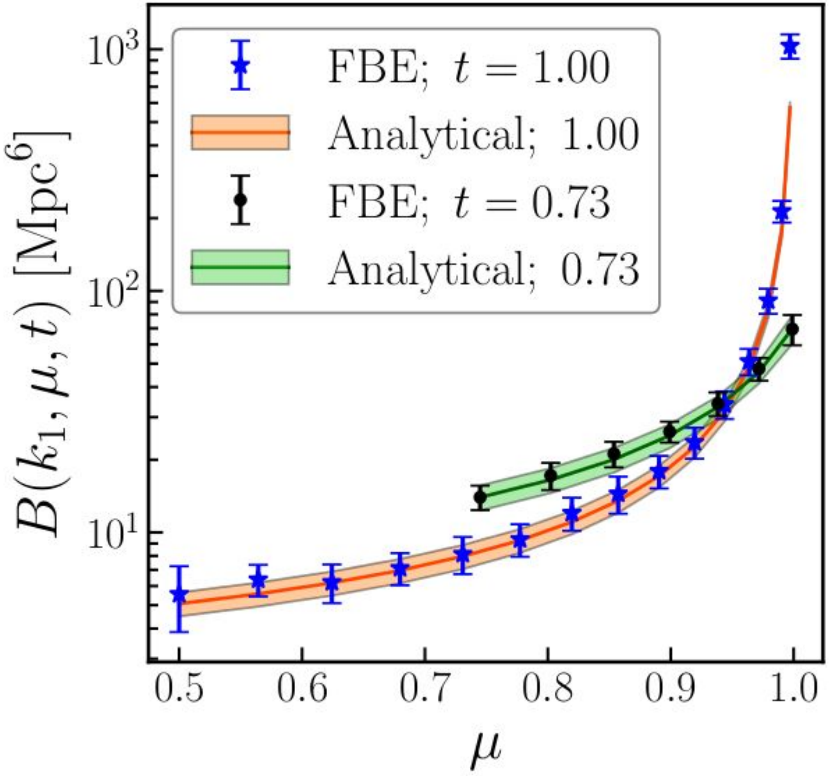
Results-I (Real space)

f_{NG}=0.5
N_g=384

Error bars are obtained from 100 realizations

Numerical predictions are consistent well within 5σ with the analytical predictions

Anomaly is found at a bin near the squeezed limit



LoS Anisotropy

- We introduce the LoS anisotropy (in Fourier space) to the previous NG field using

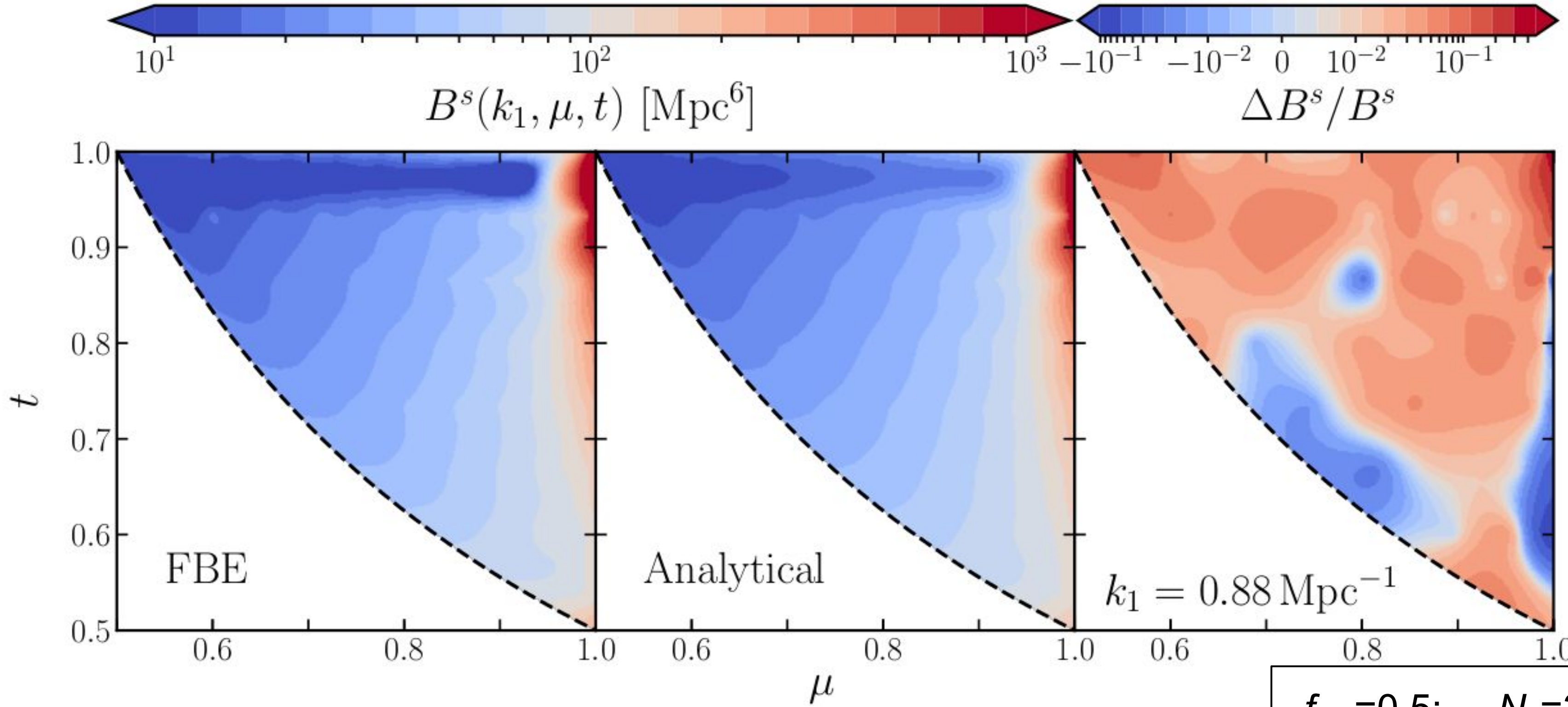
$$\Delta^s(\vec{k}) = (1 + \beta_1 \mu_1^2)\Delta^r(\vec{k}) \qquad \text{where} \qquad \beta_1 = 1$$

- Compared the FBE monopole moment estimates against the analytical predictions (for mild NG)

$$B_0^0(k_1, \mu, t) = B_{\rm Ana}(k_1, \mu, t) \times \left[1 + \beta_1 + \frac{3\beta_1^2}{5} + \frac{\beta_1^3}{7} - \frac{4\beta_1^2(3\beta_1 + 7)(1 - \mu^2)(t^2 - \mu t + 1)}{105(t^2 - 2\mu t + 1)}\right]$$

(Bharadwaj+ 20)

Results-I （Redshift space）



f_{NG}=0.5; N_g=384

Mean bispectrum computed over an ensemble of **100 realizations**

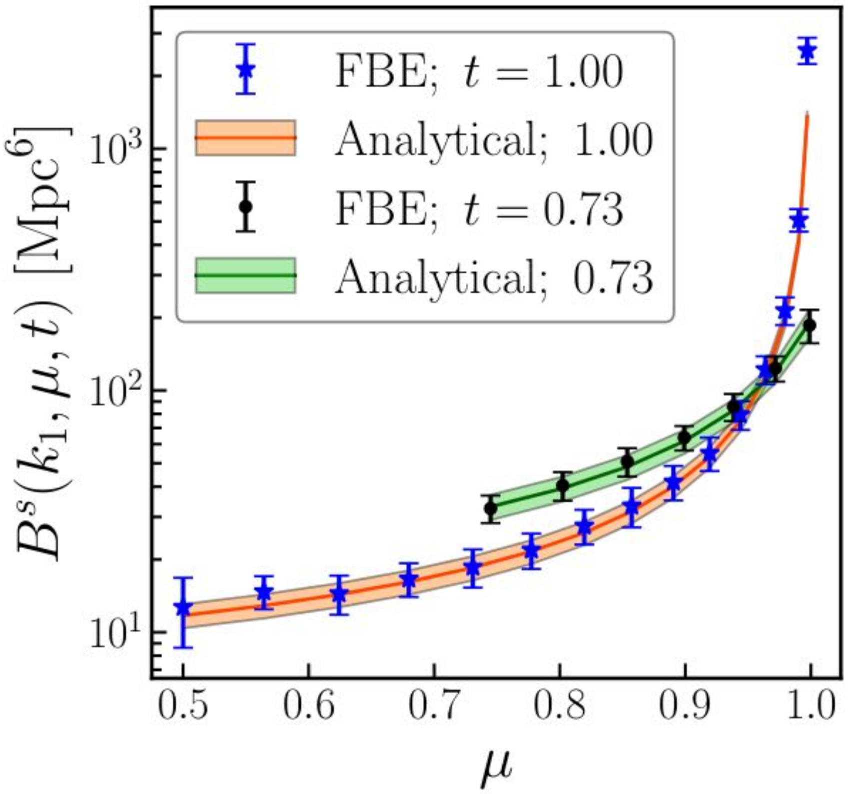
Results-I (Redshift space)

f_{NG}=0.5
N_g=384

Error bars are obtained from 100 realizations

Numerical predictions are consistent well within 5σ with the analytical predictions

Anomaly is found at a bin near the squeezed limit



Application to DM Density Field

Gravitationally-induced NG

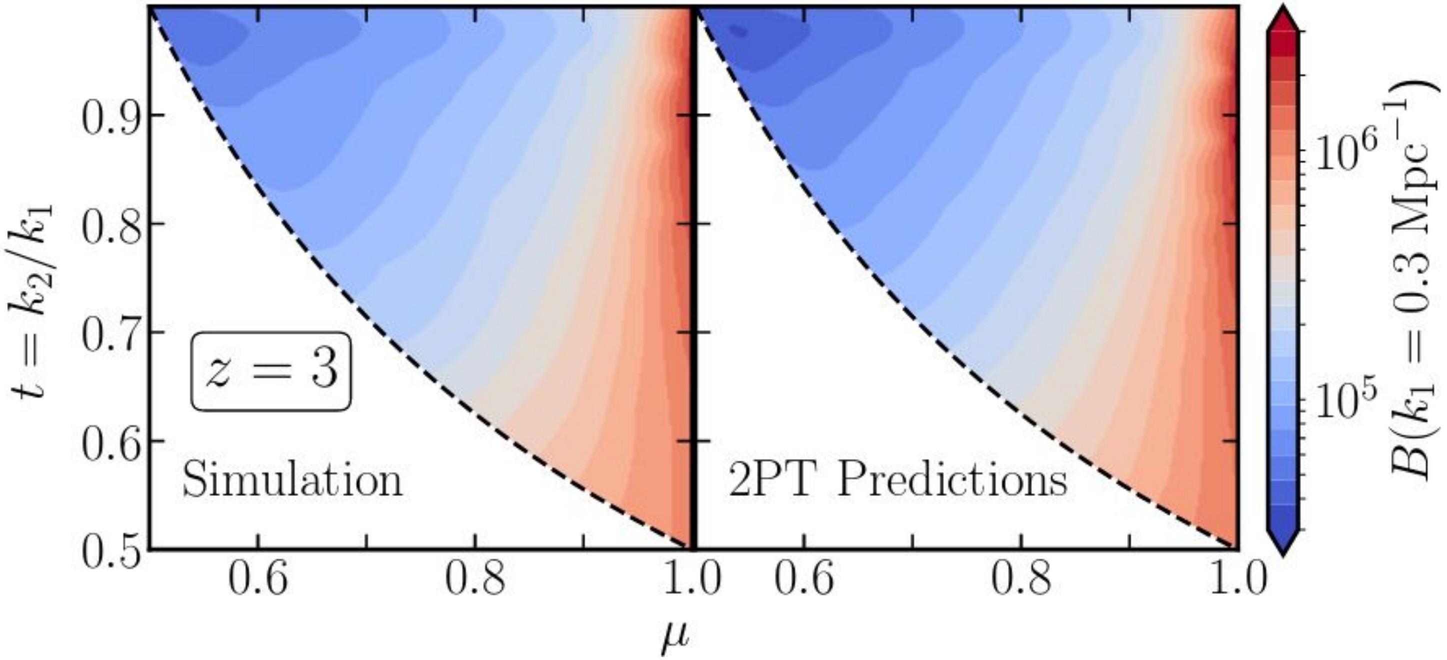
- Dark matter densities at z= 3, 2, 1. (50 realizations)
- Box volume V= [512 Mpc]^3
- Grid size --> [2 Mpc]
- Computed the bispectrum using FBE and compare it against the 2nd order perturbation theory (2PT) predictions.

(Bharadwaj+ 20)

$$B(k_1, k_2, k_3) = 2\,[F_2(\boldsymbol{k}_1, \boldsymbol{k}_2)P(k_1)P(k_2) + F_2(\boldsymbol{k}_2, \boldsymbol{k}_3)P(k_2)P(k_3) + F_2(\boldsymbol{k}_3, \boldsymbol{k}_1)P(k_3)P(k_1)]$$

$$\begin{aligned} F_{12}(\mu, t) &= \frac{1}{14}\left(4\mu^2 - 7\mu t - \frac{7\mu}{t} + 10\right) \\ F_{23}(\mu, t) &= \frac{7\mu + (3 - 10\mu^2)\,t}{14ts^2}, \\ F_{31}(\mu, t) &= \frac{t^2\left(-10\mu^2 + 7\mu t + 3\right)}{14s^2}. \end{aligned}$$

Results-II (Real Space)



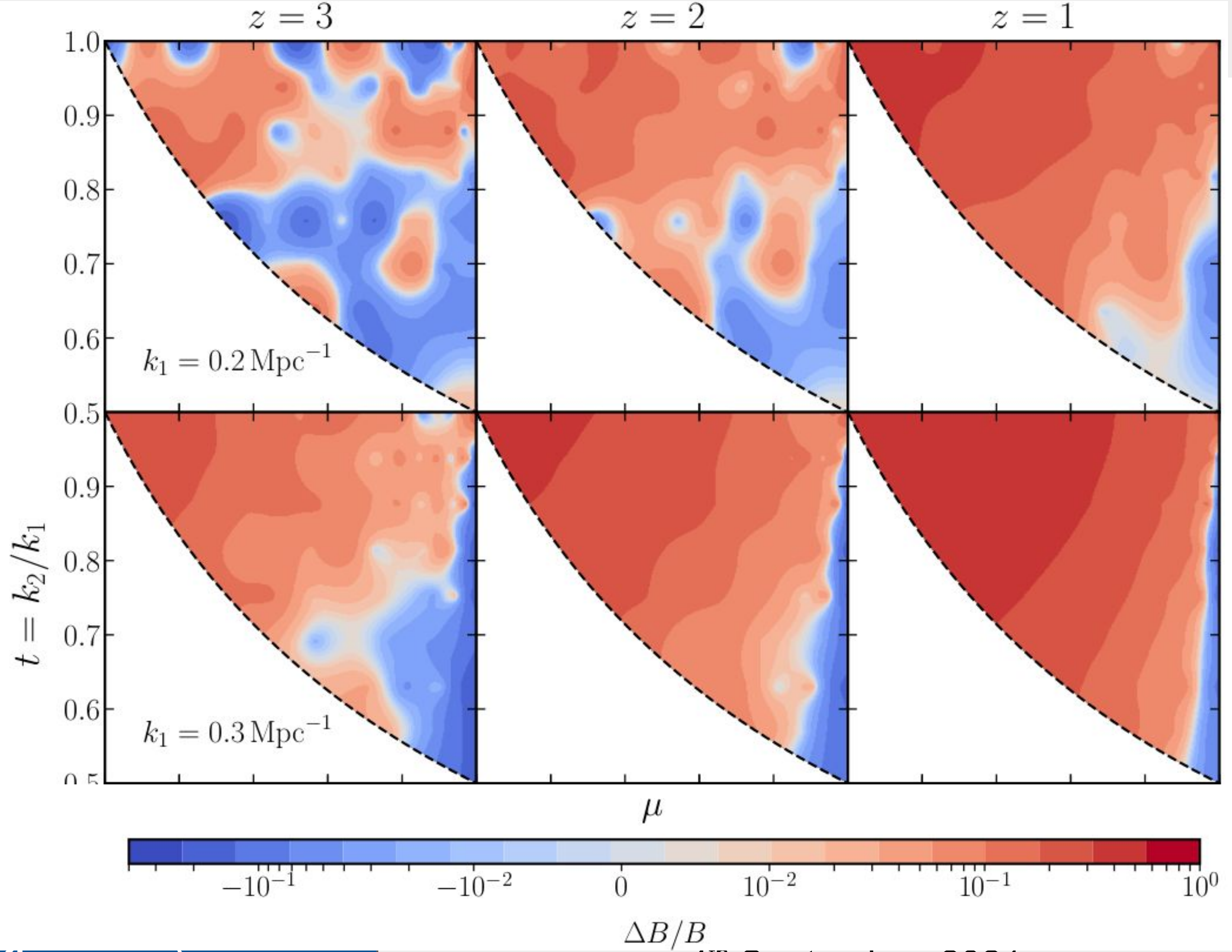
Qualitatively matches with the 2PT predictions and quantitatively close

Results-II

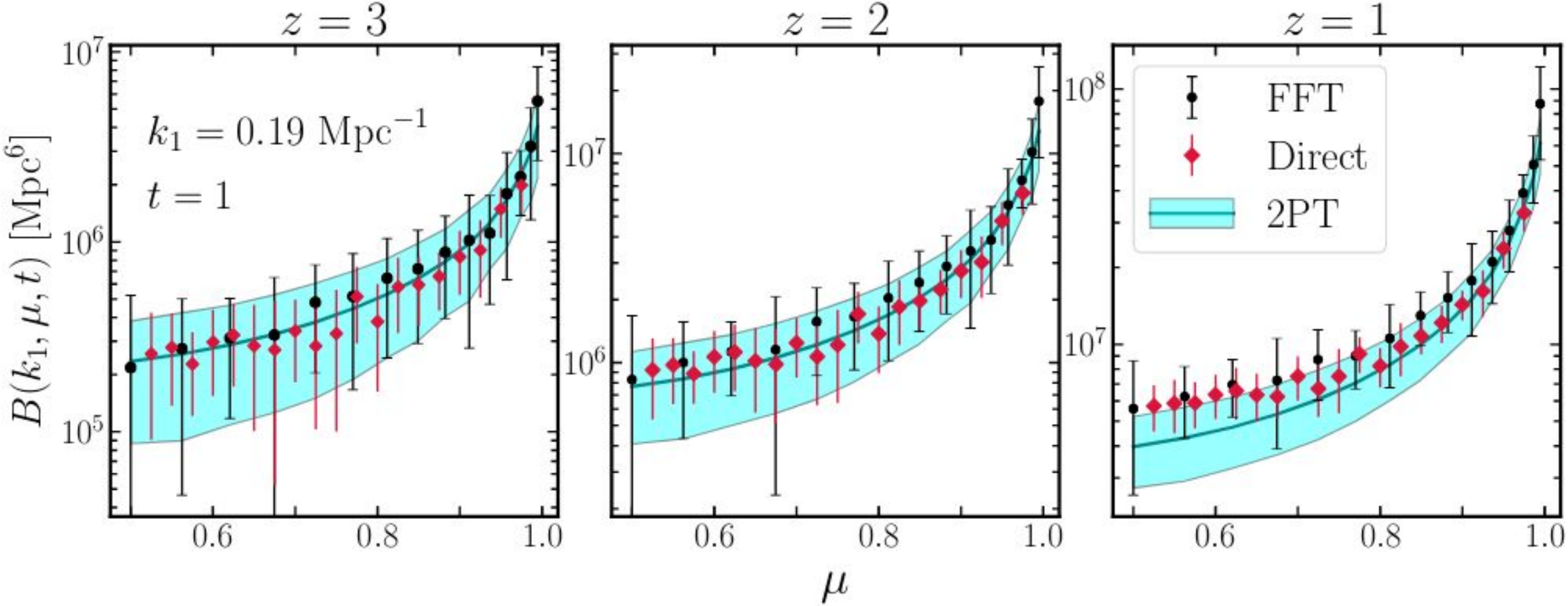
Largely dominated by cosmic variance at small k_1 values and higher z.

Fluctuations are random and small wherever 2PT is expected to be valid.

There is a significant positive deviations towards larger k_1 and lower z.



Comparing with the Direct Bisp. Estim.



Bispectra for the dark matter density field matches well with the DBE (Majumdar+ 18) within 1σ

Take Home

- The FBE has O(N^3) improvement in speed over the direct method.
- Considers all unique possible shapes of triangles
- Consistent results in case of **2nd order biased non-Gaussian fields** and also for the **dark matter density fields**. Deviations from analytical predictions are within ~ 3σ
- Can also be applied to EoR/ Post-EoR signal (Ongoing).
- With proper modifications to FBE one can also compute the different multipole moments of the bispectrum to quantify the anisotropy such as RSD effects.