New Dynamical Degrees of Freedom from Invertible Transformations

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Pavel Jiroušek, Keigo Shimada, Alexander Vikman, MY,
arXiv: 2208.05951, JHEP 07 (2023) 154.

$$c = \hbar = M_G^2 = 1/(8\pi G) = 1$$

Contents

Introduction

Field transformations are ubiquitous in mathematics & physics !!

- **Gauge (global) transformation of fields**
- **Bogliubov transformation**
- **Fourier transformation (series) & Laplace transformation**
- **Galilean, Lorentz, general coordinate transformations**
- **…**

It provides better ways to understand various physical phenomena, and to advance calculations, in particular, to solve more easily differential equations.

Conformal & disformal transformations in gravity

- **Conformal transformation :**

$$\tilde{g}_{\mu\nu} = \Omega(x)^2 g_{\mu\nu} \quad \overset{\Omega^2 \neq 0}{\Longleftrightarrow} \quad g_{\mu\nu} = \Omega(x)^{-2} \tilde{g}_{\mu\nu}$$

- **Disformal transformation :** **(Bekenstein 1992)**

$$\tilde{g}_{\mu\nu} = A(\phi, X) g_{\mu\nu} + B(\phi, X) \partial_\mu\phi\partial_\nu\phi, \quad X = -\frac{1}{2} g^{\sigma\tau}\partial_\sigma\phi\partial_\tau\phi$$

$$(\tilde{\phi} = \phi)$$

$$\Updownarrow \quad \det\left(\frac{\partial\tilde{g}_{\mu\nu}}{\partial g_{\alpha\beta}}\right) = A\left(A - A_{,X}X + 2B_{,X}X^2\right) \neq 0.$$

(N.B. No derivatives of the metrics)

$$g_{\mu\nu} = \tilde{A}(\phi, \widetilde{X})\tilde{g}_{\mu\nu} + \tilde{B}(\phi, \widetilde{X})\partial_\mu\phi\partial_\nu\phi, \quad \widetilde{X} = -\frac{1}{2}\tilde{g}^{\sigma\tau}\partial_\sigma\phi\partial_\tau\phi = \frac{X}{A - 2BX}$$

$$(\phi = \tilde{\phi})$$

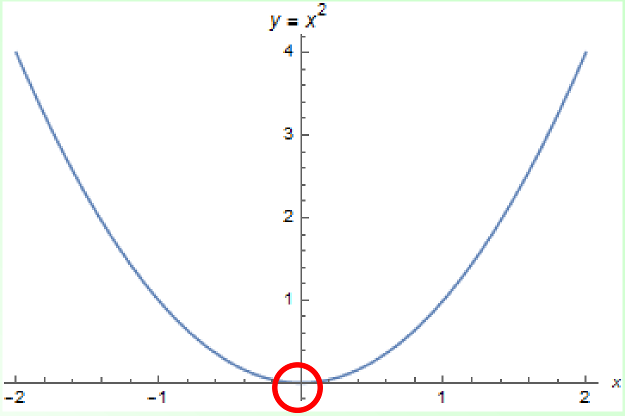
$$\left(\tilde{A}(\phi, \widetilde{X}) = \frac{1}{A(\phi, X)}, \quad \tilde{B}(\phi, \widetilde{X}) = -\frac{B(\phi, X)}{A(\phi, X)}.\right)$$

Invertible transformation

Transformation : $y_i = f_i(x_j)$

(Local) invertibility

⟺ **There is (local) one-to-one correspondence between x_j and y_i.**



The transformation $y = x^2$ is (locally) invertible except around the origin.

Except around the origin, one-to-one.

How to judge (local) invertibility ???

One way :
explicit construction of an inverse transformation

(q,φ) ← (Q,φ)

$$\begin{cases} q = Q + \phi, \\ \phi = \phi \end{cases}$$

(q,φ) → (Q,φ) ???

$$\begin{cases} Q = q - \phi, \\ \phi = \phi \end{cases}$$

⇒ **(q,φ) ⇔ (Q,φ)**

(one-to-one correspondence => invertible)

e.g. $S[q] = \frac{1}{2}\int_{t_1}^{t_2} dt\, \dot{q}^2$ ⟺ $S[Q,\phi] = \frac{1}{2}\int_{t_1}^{t_2} dt\, \left(\dot{Q} + \dot{\phi}\right)^2$

EOM: $\ddot{q} = 0$ ⟺ $\ddot{Q} + \ddot{\phi} = 0$

Nothing is changed. Just a relabeling.
The same dynamics and the same number of d.o.f.

If we cannot construct an inverse transformation concretely, how to judge ???

Regular or Singular transformation

Transformation : $y_i = f_i(x_j)$

Jacobian matrix :

$$\mathcal{J}_{ij} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

Regular ⟺ $\det \mathcal{J}_{ij} \neq 0$ ⟺ **No eigenvalue vanishes**

Singular ⟺ $\det \mathcal{J}_{ij} = 0$ ⟺ **An eigenvalue vanishes**

Inverse function theorem

Transformation : $y_i = f_i(x_j)$

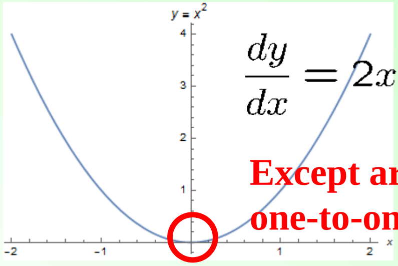
Jacobian matrix :

$$\mathcal{J}_{ij} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

Regular $\Longleftrightarrow$ $\det \mathcal{J}_{ij} \neq 0$ $\Longleftrightarrow$ **No eigenvalue vanishes**

Inverse function theorem :

If the transformation is regular at some point,
there is one-to-one correspondence (invertible) locally around that point.



$$\frac{dy}{dx} = 2x$$

Except around the origin,
one-to-one.

Consequences of regular (invertible) transformation

Dynamical system [yi] characterized by the following action :

$$S_{\rm old}[y_i] = S_{\rm old}[y_i, \dot{y}_i, \ddot{y}_i, \cdots]$$

Transformation : $y_i = f_i(x_j)$

→ **New dynamical system [xj] characterized by the following action :**

$$S_{\rm new}[x_i] = S_{\rm old}[f_i(x_j)]$$

→ **Least action principle :** $\delta S_{\rm new} = \int d^4x \, \frac{\delta S_{\rm old}}{\delta y_i} \, \mathcal{J}_{ij} \, \delta x_j$

$\mathcal{J}_{ij} \left(= \frac{\delta y_i}{\delta x_j} \right)$ **: regular (at all points)** → $\frac{\delta S_{\rm new}}{\delta x_j} = 0 \iff \frac{\delta S_{\rm old}}{\delta y_i} = 0.$

Both systems are completely equivalent (just re-labeling the dynamical variables).

One-to-one correspondence

Singular but invertible transformation

N.B. A regular and invertible transformation gives the physically same dynamics (and the same number of d.o.f.) because it is nothing but re-labelling.

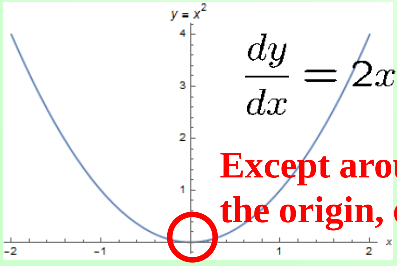
Inverse function theorem II

Transformation : $y_i = f_i(x_j)$

Jacobian matrix :

$$\mathcal{J}_{ij} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

Regular ⟺ $\det \mathcal{J}_{ij} \neq 0$ ⟺ **No eigenvalue vanishes**



Except around the origin, one-to-one.

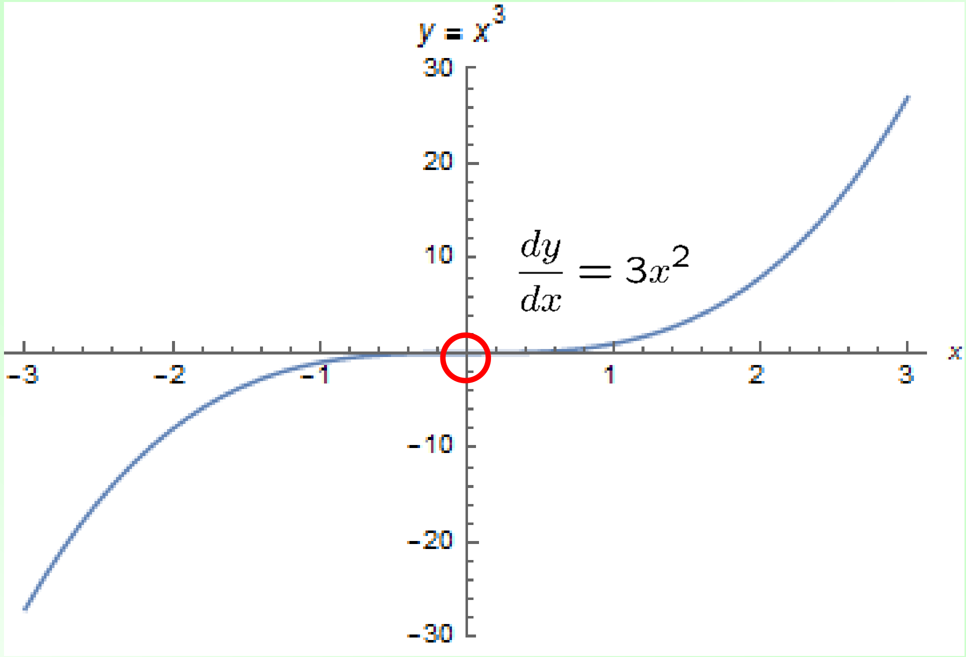
Inverse function theorem :

If the transformation is regular at some point, there is one-to-one correspondence (invertible) locally around that point.

But, the opposite is not necessarily true, that is, even if the transformation is singular at some point, it can be (locally) invertible.

Singular ⟺ $\det \mathcal{J}_{ij} = 0$ ⟺ **An eigenvalue vanishes**

Example of singular but invertible transformation



This transformation is singular at the origin.

But, the transformation is invertible everywhere!!

Regular $\Longleftrightarrow$ $\det \mathcal{J}_{ij} \neq 0$ $\Longrightarrow$ **(locally) invertible**

(Inverse function theorem)

Singular $\Longleftrightarrow$ $\det \mathcal{J}_{ij} = 0$ $\Longrightarrow$ **invertible or non-invertible**

Consequences of singular transformation

$$S_{\rm new}[x_i] = S_{\rm old}[f_i(x_j)] \qquad\qquad y_i = f_i(x_j)$$

$\Rightarrow$ $$\delta S_{\rm new} = \int d^4x \, \frac{\delta S_{\rm old}}{\delta y_i} \, \mathcal{J}_{ij} \, \delta x_j$$

(i) $\mathcal{J}_{ij} \left(= \frac{\delta y_i}{\delta x_j} \right)$ **: regular** $\Rightarrow$ $\frac{\delta S_{\rm new}}{\delta x_j} = 0 \iff \frac{\delta S_{\rm old}}{\delta y_i} = 0.$

(ii) $\mathcal{J}_{ij} \left(= \frac{\delta y_i}{\delta x_j} \right)$ **: singular (det J = 0 at some point)**

$\Rightarrow$ $\delta S_{\rm new} = 0$ **for such a singular point.**

$\Rightarrow$ $\frac{\delta S_{\rm new}}{\delta x_j} = 0 \;\not\!\!\iff\; \frac{\delta S_{\rm old}}{\delta y_i} = 0.$

New dynamics can appear for singular transformation even if it is invertible !!

Appearance of "new d.o.f" from singular but invertible transformation

$$S[q] = \frac{1}{2}\int_{t_1}^{t_2} dt\, \dot{q}^2 \quad \Rightarrow \quad \ddot{q} = 0$$

$$\Rightarrow \quad q(t) = q_0 + v_0\, t$$

(2 initial conditions: q_0 (initial position), v_0 (initial velocity) ➔ 1 d.o.f.)

(Singular but) invertible transformation of the variable Q ⇔ q :

$$\begin{cases} q = Q^3 + \dot{\phi} \\ \phi = \phi \end{cases} \quad \Leftrightarrow \quad \begin{cases} Q = \sqrt[3]{q - \dot{\phi}} \\ \phi = \phi \end{cases}$$

one-to-one correspondence
(q,φ) ⇔ (Q,φ)

$$J = \frac{\partial q}{\partial Q} = 3Q^2$$

singular at Q = 0

(The extension of the inverse function theorem to a transformation with derivatives:
Babichev, Izumi, Tanahashi, MY, 1907.12333, 2109.00912)

Appearance of new d.o.f from singular but invertible transformation II

$$q(Q,\phi) = Q^3 + \dot{\phi} \qquad \left(J = \frac{\partial q}{\partial Q} = 3Q^2 \right)$$

$$S[Q,\phi] = \frac{1}{2}\int_{t_1}^{t_2} dt \left(\ddot{\phi} + 3Q^2\dot{Q}\right)^2 \Leftarrow S[q] = \frac{1}{2}\int_{t_1}^{t_2} dt\, \dot{q}^2$$

$$\Rightarrow \frac{\delta S}{\delta Q} = -\ddot{q}\, J = 0, \qquad \frac{\delta S}{\delta \phi} = \frac{d}{dt}\ddot{q} = 0 \; .$$

- **Regular branch with J ≠ 0** ⇒ $\ddot{q} = 0$ ⇒ $q(t) = q_0 + v_0\, t$

  (2 constants, 1 d.o.f.)

- **Singular branch with J = 0 at Q = 0,**

  ⇒ $\dfrac{\delta S}{\delta \phi} = \dfrac{d^4}{dt^4}\phi = 0$

  ⇒ $\phi(t) = c_0 + c_1 t + c_2\dfrac{t^2}{2} + c_3\dfrac{t^3}{6}$

  ⇒ $q(Q,\dot{\phi})|_{Q=0} = \dot{\phi} = c_1 + c_2 t + c_3\dfrac{t^2}{2}$

4 constants, 2 d.o.f.
New d.o.f. appeared !!

Equivalent Action

$$S_1[q] = \frac{1}{2}\int_{t_1}^{t_2} dt\, \dot{q}^2 \quad \Rightarrow \quad S_2[Q,\phi] = \frac{1}{2}\int_{t_1}^{t_2} dt\, \left(\ddot{\phi} + 3Q^2\dot{Q}\right)^2$$

$$\left(q(Q,\phi) = Q^3 + \dot{\phi}\right)$$

$$\Leftrightarrow \quad S_3[Q,\phi,\theta,\lambda] = \int_{t_1}^{t_2} dt\, \left[\frac{1}{2}\left(\dot{\theta} + 3Q^2\dot{Q}\right)^2 + \lambda\left(\dot{\phi} - \theta\right)\right]$$

$$\left(\frac{\delta S}{\delta\lambda} = \dot{\phi} - \theta = 0\right)$$

gauge symmetry : $\phi \to \phi + \epsilon,\ Q^3 \to Q^3 - \dot{\epsilon},\ \theta \to \theta + \dot{\epsilon}$

$\Rightarrow$ **gauge invariant variable :** $\bar{q} = Q^3 + \theta$

$$\Leftrightarrow \quad S_4[Q,\phi,\bar{q},\lambda] = \int_{t_1}^{t_2} dt\, \left[\frac{1}{2}\dot{\bar{q}}^2 + \lambda\left(\dot{\phi} + Q^3 - \bar{q}\right)\right] \Longrightarrow \int_{t_1}^{t_2} dt\, \left[\frac{1}{2}\ddot{\phi}^2\right]$$

$$\Rightarrow \quad \frac{\delta S}{\delta Q} = 3\lambda Q^2 = 0$$

- **Regular branch with λ = 0** $\Rightarrow$ $S_1[\bar{q}]$
- **Singular branch with Q = 0** $\Rightarrow$ $\bar{q} = \dot{\phi}$

Yet another interesting example

N.B. A regular and invertible transformation gives the physically same dynamics (and the same number of d.o.f.) because it is nothing but re-labelling.

Appearance of new "dynamics" from singular but invertible transformation

$$S[q,\phi] = \frac{1}{2}\int_{t_1}^{t_2} dt\, \left(\dot{q}^2 + \dot{\phi}^2\right) \Longrightarrow \ddot{q} = 0,\ \ \ddot{\phi} = 0.$$

$$\Longrightarrow q(t) = q_0 + v_0\, t,\ \ \phi(t) = \phi_0 + u_0\, t.$$

(**4 initial conditions**: q_0, v_0, φ_0, u_0 ➔ **2 d.o.f.**)

(Singular but) invertible transformation of the variables :

$$\begin{cases} q = Q^3 + \dot{\phi}\ , \\ \phi = \phi\ . \end{cases}$$

Appearance of new “dynamics” from singular but invertible transformation II

$$q = Q^3 + \dot{\phi}\ , \ \phi = \phi\ .$$

$$S\left[Q,\phi\right] = \frac{1}{2}\int_{t_1}^{t_2} dt\ \left[\left(\ddot{\phi} + 3Q^2\dot{Q}\right)^2 + \dot{\phi}^2\right] \Leftarrow S[q,\phi] = \frac{1}{2}\int_{t_1}^{t_2} dt\ \left(\dot{q}^2 + \dot{\phi}^2\right)$$

$$\Rightarrow \frac{\delta S}{\delta Q} = -3\ddot{q}Q^2 = 0\ , \quad \frac{\delta S}{\delta \phi} = \frac{d}{dt}\left(\ddot{q} - \dot{\phi}\right) = 0\ .$$

- **Regular branch with Q ≠ 0** ⇒ $\ddot{q} = 0,\ \ddot{\phi} = 0.$ **(4 constants, 2 d.o.f.)**
- **Singular branch with Q = 0,**

$$\Rightarrow \frac{\delta S}{\delta \phi} = \frac{d}{dt}\left(\dddot{\phi} - \dot{\phi}\right) = 0$$

$$\Rightarrow \phi(t) = c_0 + c_1 t + c_2 \sinh(t) + c_3 \cosh(t)$$

$$\Rightarrow q(Q,\dot{\phi})|_{Q=0} = \dot{\phi} = c_1 + c_2 \sinh(t) + c_3 \cosh(t)$$

4 constants, 2 d.o.f.
The number of d.o.f. remains unchanged.
But, “new dynamics” appeared!!

Mimetic gravity example

Mimetic gravity (dark matter)

(Chamseddine & Mukhanov 2013)

Seed system : **$g_{\mu\nu}$ and matter field (Ψ_M) with a seed action,**

$$S_{\text{seed}}[g, \Psi_M]$$

Singular transformation (with conformal (Weyl) invariance)

$$g_{\mu\nu} = g_{\mu\nu}(h_{\sigma\tau}, \phi) \quad \left(g_{\mu\nu} \to g_{\mu\nu} \text{ for } h_{\mu\nu} \to \omega^2 h_{\mu\nu}\right)$$

Transformed system : **$h_{\mu\nu}$ and matter field (Ψ_M) with**

new d.o.f (φ) (constrained by conformal inv.)

$$S_{\text{dis}}[h, \phi, \Psi_M] = S_{\text{seed}}[g(h, \phi), \Psi_M]$$

$$\Rightarrow \quad \frac{\delta S_{\text{dis}}[h, \phi, \Psi_M]}{\delta h_{\mu\nu}} = 0, \quad \frac{\delta S_{\text{dis}}[h, \phi, \Psi_M]}{\delta \phi} = 0,$$

give gravitational eq. including new d.o.f and its eq.

Concrete (original) example of mimetic gravity

(Chamseddine & Mukhanov 2013)

Seed system : $g_{\mu\nu}$ and matter field (Ψ_M) with a seed action,

e.g. $$S_{\rm seed}[g,\Psi_M] = \int d^4x\sqrt{-g}R + S_{\rm matter}$$

Singular transformation with conformal invariance ⬇

$$g_{\mu\nu} = \left(h^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right)h_{\mu\nu} \equiv Y h_{\mu\nu}$$

$$\left(g_{\mu\nu}\to g_{\mu\nu}\ \text{for}\ h_{\mu\nu}\to\omega^2 h_{\mu\nu}\right)$$

(Dis)transformed system : $h_{\mu\nu}$ and matter field (Ψ_M) with

new d.o.f φ (constrained by conformal inv.)

$$S_{\rm dis}[h,\phi,\Psi_M] = S_{\rm seed}[g(h,\phi),\Psi_M]$$

$$\begin{cases} \delta S_{\rm seed}[g,\Psi_M] = \int d^4x\sqrt{-g}\left(G^{\mu\nu}(g)-T^{\mu\nu}\right)\delta g_{\mu\nu} \\ \delta g_{\mu\nu} = Y\delta h_{\sigma\rho}\left(\delta^\sigma_\mu\delta^\rho_\nu - g_{\mu\nu}g^{\sigma\alpha}g^{\rho\beta}\partial_\alpha\phi\partial_\beta\phi\right) + 2g_{\mu\nu}g^{\sigma\rho}\partial_\sigma\delta\phi\partial_\rho\phi \end{cases}$$

➡ $$\frac{\delta S_{\rm dis}[h,\phi,\Psi_M]}{\delta h_{\mu\nu}} = 0, \quad \frac{\delta S_{\rm dis}[h,\phi,\Psi_M]}{\delta\phi} = 0$$

Concrete (original) example of mimetic gravity II

(Chamseddine & Mukhanov 2013)

$$S_{\rm dis}[h,\phi,\Psi_M] = S_{\rm seed}[g(h,\phi),\Psi_M] = \int d^4x\sqrt{-g}R + S_{\rm matter}.$$

$$g_{\mu\nu} = \left(h^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right)h_{\mu\nu} \equiv Y h_{\mu\nu} \qquad \left(g_{\mu\nu}\to g_{\mu\nu} \text{ for } h_{\mu\nu}\to\omega^2 h_{\mu\nu}\right)$$

$$\left(\Longrightarrow g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = \frac{h^{\mu\nu}}{Y}\partial_\mu\phi\partial_\nu\phi = 1\right) \qquad \underline{(G(g)-T)}\left(1 - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right) = 0.$$

$\neq 0$ **Trace part is trivially satisfied.**

$$\delta S_{\rm seed}[g,\Psi_M] = \int d^4x\sqrt{-g}\left(G^{\mu\nu}(g) - T^{\mu\nu}\right)\delta g_{\mu\nu}$$

$$\delta g_{\mu\nu} = Y\delta h_{\sigma\rho}\left(\delta^\sigma_\mu\delta^\rho_\nu - g_{\mu\nu}g^{\sigma\alpha}g^{\rho\beta}\partial_\alpha\phi\partial_\beta\phi\right) + 2g_{\mu\nu}g^{\sigma\rho}\partial_\sigma\delta\phi\partial_\rho\phi$$

$$\frac{\delta S_{\rm dis}[h,\phi,\Psi_M]}{\delta h_{\mu\nu}} = 0 \quad\Longrightarrow\quad G^{\mu\nu}(g) - T^{\mu\nu} - (G(g)-T)g^{\mu\sigma}g^{\nu\rho}\partial_\sigma\phi\partial_\rho\phi = 0$$

$$\Longleftrightarrow\quad G^{\mu\nu}(g) = T^{\mu\nu} + \tilde{T}^{\mu\nu} \quad \textbf{with} \quad \overset{(g)}{\nabla}_\mu \tilde{T}^\mu_\nu = 0$$

Fix ε

$$\frac{\delta S_{\rm dis}[h,\phi,\Psi_M]}{\delta\phi} = 0 \quad\Longrightarrow\quad \overset{(g)}{\nabla}_\alpha\left((G(g)-T)\partial^\alpha\phi\right) = 0$$

$$\tilde{T}^{\mu\nu} \equiv (\epsilon+p)u^\mu u^\nu - pg^{\mu\nu}, \quad \epsilon = G(g) - T,\ p = 0,\ u^\mu = g^{\mu\alpha}\partial_\alpha\phi \quad \textbf{(dust)}$$

$$(u^\mu u_\mu = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 1)$$

What's the essence of mimetic gravity ?

(Pavel Jiroušek, Keigo Shimada, Alexander Vikman, MY, arXiv: 2207.12611, JCAP 11 (2022) 019)

Singular
(disformal) transformation

Disformal transformation

(Bekenstein 1992)

$$g_{\mu\nu} = C(Y,\phi)h_{\mu\nu} + D(Y,\phi)\partial_\mu\phi\partial_\nu\phi, \quad Y = h^{\sigma\tau}\partial_\sigma\phi\partial_\tau\phi$$

$$\left(g^{\mu\nu} = \frac{1}{C}\left(h^{\mu\nu} - \frac{D}{C+DY}\partial^\mu\phi\partial^\nu\phi \right) \right)$$

To check the invertibility, we consider Jacobian, its eigenvalues and eigenvectors.

(Zumalacárregui & García-Bellido 2014)

$$\mathcal{J}^{\mu\nu}_{\sigma\rho} = \frac{\delta g_{\sigma\rho}}{\delta h_{\mu\nu}} = C\delta^\mu_\sigma\delta^\nu_\rho - C_Y h_{\sigma\rho}\partial^\mu\phi\partial^\nu\phi - D_Y\partial_\sigma\phi\partial_\rho\phi\partial^\mu\phi\partial^\nu\phi$$

$$\mathcal{J}^{\mu\nu}_{\sigma\rho}\xi^a_{\mu\nu} = \lambda_a\xi^a_{\sigma\rho}, \quad \zeta_a^{\sigma\rho}\mathcal{J}^{\mu\nu}_{\sigma\rho} = \lambda_a\zeta_a^{\mu\nu}, \qquad \left(C_Y \equiv \frac{\partial C}{\partial Y},\ D_Y \equiv \frac{\partial D}{\partial Y} \right)$$

- **(9) eigenvalues, eigenvectors, dual-eigenvectors :**

$$\lambda_C = C, \quad \xi^C_{\mu\nu} = \phi^\perp_{\mu\nu}, \quad \zeta_C^{\mu\nu} = \phi_\top^{\mu\nu} \qquad \left(\phi^\perp_{\mu\nu}\partial^\mu\phi\partial^\nu\phi = 0, \quad \phi_\top^{\mu\nu}\xi^D_{\mu\nu} = 0 \right)$$

- **(1) eigenvalue, eigenvector, dual-eigenvector :**

$$\lambda_D = C - C_Y Y - D_Y Y^2, \quad \xi^D_{\mu\nu} = C_Y h_{\mu\nu} + D_Y\partial_\mu\phi\partial_\nu\phi, \quad \zeta_D^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi$$

$\lambda_C = 0$ and/or $\lambda_D = 0$ ⇔ Singular transformation

Consequences of singular transformation

$$S_{\text{dis}}[h,\phi,\Psi_M] = S_{\text{seed}}[g(h,\phi),\Psi_M]$$

$$\Rightarrow \quad \delta S_{\text{dis}} = \int d^4x \, \frac{\delta S_{\text{seed}}}{\delta g_{\sigma\rho}} \mathcal{J}^{\mu\nu}_{\sigma\rho} \, \delta h_{\mu\nu}$$

(i) $\mathcal{J}^{\mu\nu}_{\sigma\rho} \left(= \frac{\delta g_{\sigma\rho}}{\delta h_{\mu\nu}} \right)$ **: regular** $\Rightarrow \quad \frac{\delta S_{\text{dis}}}{\delta h_{\mu\nu}} = 0 \iff \frac{\delta S_{\text{seed}}}{\delta g_{\sigma\rho}} = 0.$

(ii) $\mathcal{J}^{\mu\nu}_{\sigma\rho} \left(= \frac{\delta g_{\sigma\rho}}{\delta h_{\mu\nu}} \right)$ **: singular (λa = 0)**

$$\Rightarrow \quad \delta S_{\text{dis}} = \int d^4x \left(\frac{\delta S_{\text{seed}}}{\delta g_{\sigma\rho}} - \rho \zeta_a^{\sigma\rho} \right) \mathcal{J}^{\mu\nu}_{\sigma\rho} \, \delta h_{\mu\nu}$$

$\left(\frac{\delta S_{\text{dis}}}{\delta h_{\mu\nu}} = 0 \right)$ $\Rightarrow \quad \frac{\delta S_{\text{seed}}}{\delta g_{\sigma\rho}} = \rho \zeta_a^{\sigma\rho}$

$$\left(\zeta_a^{\sigma\rho} \mathcal{J}^{\mu\nu}_{\sigma\rho} = \lambda_a \zeta_a^{\mu\nu}, \ \lambda_a = 0 \right)$$

In original case (C=Y≠0, D=0)

$$\Rightarrow \quad G_{\mu\nu} - T_{\mu\nu} = \tilde{\rho} \, \partial_\mu \phi \partial_\nu \phi = \tilde{T}_{\mu\nu} \qquad \left(\tilde{\rho} = \frac{(C+DY)^2}{\sqrt{-g}} \rho \right)$$

$\left(\lambda_D = C - C_Y Y - D_Y Y^2 = 0 \right)$

|| **(G – T)**

The important message :

The property of mimetic matter
is determined by the (dual) eigenvector with
zero eigenvalue of
a "singular" transformation.

⇩

It has been supposed that non-invertibility of
a transformation is inevitable for mimetic gravity.
But, this is not the case, as we will show.

Singular behavior of disformal transformation

$$g_{\mu\nu} = C(Y,\phi)h_{\mu\nu} + D(Y,\phi)\partial_\mu\phi\partial_\nu\phi, \qquad Y = h^{\sigma\tau}\partial_\sigma\phi\partial_\tau\phi$$

Jacobian matrix, its eigenvalues and eigenvectors.

$$\mathcal{J}^{\mu\nu}_{\sigma\rho} = \frac{\delta g_{\sigma\rho}}{\delta h_{\mu\nu}} = C\delta^\mu_\sigma\delta^\nu_\rho - C_Y h_{\sigma\rho}\partial^\mu\phi\partial^\nu\phi - D_Y\partial_\sigma\phi\partial_\rho\phi\partial^\mu\phi\partial^\nu\phi\,, \qquad \mathcal{J}^{\mu\nu}_{\sigma\rho}\xi^a_{\mu\nu} = \lambda_a\xi^a_{\sigma\rho},$$

(1) eigenvalue, eigenvector:

$$\lambda_D = C - C_Y Y - D_Y Y^2 = -Y^2\partial_Y\left(\frac{C}{Y} + D\right), \quad \xi^D_{\mu\nu} = C_Y h_{\mu\nu} + D_Y\partial_\mu\phi\partial_\nu\phi,$$

- **$\lambda_D = 0$ as a function (for all configurations of φ)** (Deruelle & Rua 2014)

  ⇕ $$D(Y,\phi) = -\frac{C(Y,\phi)}{Y} + c(\phi)$$ ← **arbitrary function**

- **$\lambda_D = 0$ for some configuration for φ** ← **this talk**

  ⇕ $$C = C_Y Y + D_Y Y^2$$

interpreted as a non-trivial equation of motion which may be used to determine the behavior of φ.

Concrete example of mimetic gravity with "invertible" transformation

$$g_{\mu\nu} = C(Y,\phi) h_{\mu\nu}, \quad C(Y,\phi) = \frac{Y}{(Y-1)^3+1} \geq 0, \qquad \left(Y = h^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right)$$

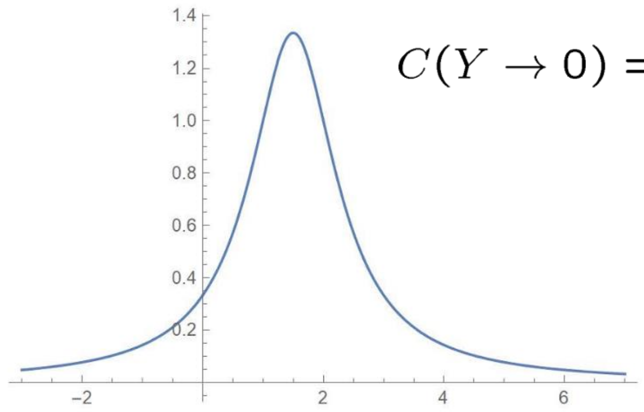


Figure 1. Graph of the function $C(Y)$ given by equation (4.9). The maximum lies at $Y = 3/2$.

Singular at Y = 1 : $C - C_Y Y = -\frac{3Y^2}{((Y-1)^3+1)^2}(Y-1)^2$

But, invertible : $h_{\mu\nu} = \frac{X}{\sqrt[3]{X-1}+1} g_{\mu\nu}$

Summary

- Transformation is ubiquitous.
- Among them, invertible transformation is special because there is one-to-one correspondence between old and new variables.
- For regular transformation with its Jacobian being non-vanishing, the inverse function theorem guarantees its (local) invertibility.
- Two dynamical systems are completely equivalent if two systems are connected through regular (invertible) transformation.
- Singular transformation can be invertible or non-invertible.
- Singular but invertible can change d.o.f as well as change dynamics.
- A concrete mimetic example with a singular but invertible transformation is given.