The Matter Power Spectrum

A project report
submitted in partial fulfillment for the award of the degree of
Bachelor of Technology
in
Engineering Physics
by

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under the guidance of
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CERTIFICATE

This is to certify that the project titled **The Matter Power Spectrum** is a bona fide record of work done by **S. Sindhu Sri Sravya** towards the partial fulfillment of the requirements of the Bachelor of Technology degree in Engineering Physics at the Indian Institute of Technology, Madras, Chennai 600036, India.

(L. Sriramkumar, Project supervisor)

ACKNOWLEDGEMENTS

I would take this opportunity to thank IIT Madras for providing us with this great opportunity. I am grateful to **Dr. L. Sriramkumar** for his constant guidance and motivation. Without his inspiration, this work wouldn't have been possible. I am grateful to my parents for always encouraging me and the support of my friends has provided me with an enjoyable environment to work in.

ABSTRACT

The main aim of this thesis is to understand the theory of structure formation in the universe and examine the quantities describing it. We study the evolution of inhomogeneities in the universe using the linear cosmological perturbation theory. We analytically study the behaviour of gravitational potential and matter density as the universe evolves from radiation-dominated epoch to matter-dominated epoch. We then derive quantities measurable from observations, including the matter power spectrum and compare it to the power spectrum obtained from galaxy clustering observations of SDSS.

Contents

Chapter 1

Introduction

1.1 The Smooth Universe

1.1.1 The Cosmological Principle

For centuries, astronomers speculated as to whether we are in a privileged location with respect to the rest of the universe. The general assumption is that there is nothing special about our location in the universe and it is known as the Copernican principle. Based on this, one can conclude that if we observe the universe around us to be isotropic, it can be assumed that the universe is homogeneous and isotropic everywhere, since isotropy at every point in space implies homogeneity.This is the basis for modern Cosmological models. The Cosmological Principle is that the universe is homogeneous and isotropic when viewed on large distance scales. What it means is that statistical properties such as the average density in a region of extent more than 100Mpc is independent of where the region is located in the universe.

1.1.2 Friedmann-Lemaître-Robertson-Walker metric

Consider the most general 4 dimensional spacetime line element and apply the cosmological principle to it.

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1.1}$$

If we consider space to be isotropic, then the metric should not consist of cross terms such as $dx^i dx^j$ and $dt dx^i$ (where i,j run through space components)since it is not invariant under

spatial inversion. This simplifies the metric to

$$ds^2 = g_{00}dt^2 + d\mathbf{x}^2 \tag{1.2}$$

where **x** refers to spatial coordinates.The general g_{ij} components can be function of both space and time. The spatial line element can be written based on the fact that space is homogeneous and isotropic. The time component of spatial metric is given by the scale factor a(t), since we know that the universe is expanding.(from Hubble's law). Also, the Einstein's equations do not have a stable solution for the static line element.
The FLRW metric is given as(using the negative sign convention) :

$$ds^2 = dt^2 - a(t)^2(\frac{dr^2}{1-kr^2} + r^2 d\Omega_2^2) \tag{1.3}$$

where k is the spatial curvature, which is 0,± 1 for flat, closed and open universe respectively. $d\Omega_2^2$ is the line element on 2-sphere which is equal to $d\theta^2 + sin^2\theta d\phi^2$

1.1.3 Stress Energy Tensor

On cosmological scales, the universe can be assumed to be filled with a perfect fluid. In the comoving frame(which expands at along with the expansion of the universe), the stress energy tensor can be written as

$$T_{ij} = diag(\rho(t), -p(t), -p(t), -p(t)) \tag{1.4}$$

where ρ is the mean mass density and p is pressure of the fluid. Both ρ and p are functions of time alone, due to the homogeneity and isotropy of space.

1.2 Friedmann Equations

The Einstein's equations for the FLRW metric with the source as a perfect fluid are given by

$$(\frac{\dot{a}}{a})^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho \tag{1.5}$$

$$\frac{2\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 + \frac{k}{a^2} = -8\pi G p \tag{1.6}$$

where $\dot{a}$ represents time derivative
Combining the above two equations, we get

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{1.7}$$

where H is the Hubble parameter, defined as $\frac{\dot{a}}{a}$

1.2.1 Equation of State

So far, we have 3 parameters ρ, p, a and two equations. In order to solve for the scale factor, we shall assume the equation of state as

$$p = \rho w \tag{1.8}$$

where w is equal to 0 for non-relativistic pressure-less matter/dust, 1/3 for relativistic matter/radiation and -1 for the cosmological constant.
Substituting the equation of state and solving the above equation, we have

$$\rho \propto a^{-3(1+w)} \tag{1.9}$$

$$\implies \rho \propto a^{-3} \quad : \quad matter \tag{1.10}$$

$$\rho \propto a^{-4} \quad : \quad radiation \tag{1.11}$$

$$\rho = const. \quad : \quad dark\,energy/\ cosmological\,constant \tag{1.12}$$

1.2.2 Scale factor in different epochs

Rewriting the first Friedmann equation while accounting for all types of matter, we have :

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}(\rho^0_{NR}(\frac{a_0}{a})^3 + \rho^0_R(\frac{a_0}{a})^4 + \rho^0_\Lambda) \tag{1.13}$$

For the rest of the discussion, we shall assume that the universe is flat i.e; spatial curvature k = 0 , which is a fair assumption when compared to observations. In that case, as we see , among the three terms on the right hand side, different terms dominate depending on the value of scale factor. Let us also assume that the scale factor at the Big Bang is equal to zero.

Radiation dominated epoch

For small values of the scale factor, the radiation density dominates and the other two terms can be ignored. Solving the equation(1.13) in that case gives us:

$$a(t) \propto t^{(1/2)} \tag{1.14}$$

Matter dominated epoch

At later times after the radiation dominated era, the matter density starts dominating radiation density with increasing a(t). In that case, solving for scale factor in a matter dominated era gives us:

$$a(t) \propto t^{(2/3)} \tag{1.15}$$

Chapter 2

Linear Cosmological Perturbation Theory

2.1 The inhomogeneous universe

In the previous chapter, we discussed how the universe can be treated as homogeneous and isotropic , which makes it easier for us to study the dynamics of the universe. However, when we look around us, we find that the galaxies in the universe are not homogeneously distributed but rather exist in clusters and superclusters. Fig(2.1) below shows the distribution of galaxies as a function of redshift(or equivalently distance; a redshift of 0.01 corresponds to a distance of 30Mpc, using Hubble's law and assuming Hubble constant to be $100kms^{-1}Mpc^{-1}$). It can be seen that there are several regions which are more dense compared to the average density while some of the regions have little to no matter. These regions are also referred to as Voids. To explain why the universe is so structured on small scales, we need a theory of structure formation in the universe.
Moreover, the cosmic microwave background, as shown in Fig(2.2), is inhomogeneous upto 1 part in 10^5 meaning that the inhomogeneities would have been much smaller initially.

2.2 Perturbations in metric

Since the CMB anisotropies are of the order much lesser than 1, the initial perturbations must be of lesser order and hence can be studied using the linear perturbation theory. The perturbations in metric can be categorized into scalar, vector and tensor depending on how they behave under a coordinate transformation. The sources of scalar perturbation include perturbations in energy/mass density , while vector perturbations arise due to rotational ve-

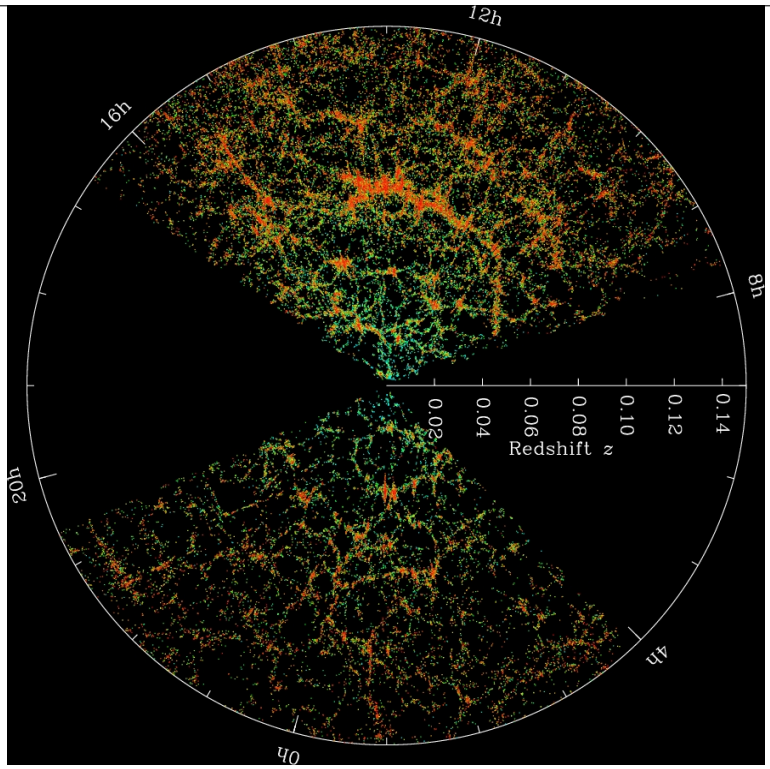

Figure 2.1: SDSS Galaxy map. Image Credit: M. Blanton and SDSS [1]

locity fields. Tensor perturbations describe gravitational waves and can exist even without source.

2.2.1 Degrees of Freedom

The metric is symmetric and therefore in D+1 dimensions, there are (D+2)(D+1)/2 components. Also, we have choice in coordinate system :

$$g_{\mu\nu} = \frac{\partial \tilde{x}^{\alpha}}{\partial x^{\mu}} \frac{\partial \tilde{x}^{\beta}}{\partial x^{\nu}} \tilde{g}_{\alpha\beta} \tag{2.1}$$

which leads to $(D+1)$ degrees of freedom. Therefore, the total degrees of freedom available for the metric will be $(D+1)(D+2)/2 - (D+1) = D(D+1)/2$.

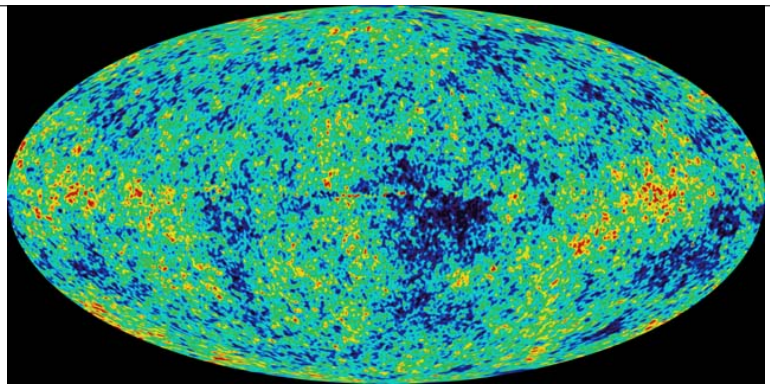

Figure 2.2: WMAP sky map showing Temperature fluctuations. Image Credit: NASA / WMAP Science Team [2]

2.2.2 S-V-T decomposition

The metric perturbation $\delta g_{\mu\nu}$ can be written in the form of scalar, vector and tensor components in a matrix form as :

$$\begin{bmatrix} \delta g_{00} & \delta g_{0i} \\ \delta g_{i0} & \delta g_{ij} \end{bmatrix} \tag{2.2}$$

where the 0 index corresponds to time component and i,j's correspond to spatial components. The scalar perturbation can be expressed as $\delta g_{00} = A$. Using Helmholtz decomposition, the vector perturbation can be written as $\delta g_{0i} = \partial_i B + \nabla C_i$, where B is a scalar and C is a divergence-less vector. Similarly , the tensor component can be decomposed as :

$$\delta g_{\mu\nu} = D\delta_{ij} + (\nabla_i E_j + \nabla_j E_i) + [(\frac{1}{2})(\nabla_i \nabla_j) - (\frac{1}{3})\delta_{ij}\nabla^2]G + H_{ij} \tag{2.3}$$

where H_{ij} is a traceless,symmetric and transverse tensor, satisfying $\nabla_i H_{ij} = 0$. We have 4 scalars A,B,D,E and divergence-free vectors C,F which have 2(D-1) degrees of freedom. The tensor H_{ij} has (D)(D+1)/2 - (D+1) = (D+1)(D-2)/2 . Therefore, the total number of degrees of freedom will be

$$4 + 2(D-1) + (D+1)(D-2)/2 = (D+1)(D+2)/2 \tag{2.4}$$

Including the degrees of freedom associated with coordinate transformation, we have $(D+1)(D+2)/2 - (D+1) = D(D+1)/2$ which we derived in the previous section.
Upto linear order, the equations for scalar, vector and tensor perturbations separate out and can be solved independently.

2.2.3 Choice of gauge

We now have to choose 'good' coordinates to describe the scalar, vector and tensor perturbations described in (2.2.2). A 'good' coordinate system is the one in which the perturbations are not spurious i.e; those which can be eliminated by a coordinate transformation. One way is to choose these perturbations such that they don't change under a coordinate transformation. These class of variables are called Bardeen variables and are given by :

$$\psi = A + H(B - G') + (B - G')' \tag{2.5}$$

$$\phi_i = F_i' - C_i \tag{2.6}$$

$$\phi = -D - H(B - G') + \frac{1}{3}\nabla^2 G \tag{2.7}$$

where primed derivatives are with respect to conformal time.
The other way easier way is to stick to a particular coordinate system and derive all equations in it.

Newtonian Gauge
Choose A = ψ and D = -ϕ and set rest of the variables to zero, so that we have the only non-zero metric perturbations to also be coordinate inavariant i.e; the Bardeen variables shown in the last section. The line element is given as

$$ds^2 = a^2(\tau)[(1 + 2\psi)d\tau^2 - (1 - 2\phi)(dx^2 + dy^2 + dz^2)] \tag{2.8}$$

Observe how the metric now is close to the weak field limit of GR. We'll see that ϕ actually corresponds to the Newtonian gravitational potential and therefore this gauge is called the Newtonian gauge.

2.2.4 Perturbed Christoffel symbols and Curvature Tensor

From the metric, the perturbed Christoffel symbols and Curvature can be calculated. Going through the tedious calculation, we find that the non-zero Christoffel symbols are:

$$\Gamma^0_{00} = a\phi'$$

$$\Gamma^0_{0i} = \frac{\partial_i \phi}{1 + 2\phi} = \partial i \phi$$

$$\Gamma^0_{ij} = \delta_{ij}[\frac{a'(1-2\psi) - a\psi'}{1+2\phi}] = [a'(1-2\phi-2\psi) - a\psi']\delta_{ij}$$

$$\Gamma^i_{00} = \frac{\partial_i\psi}{1-2\psi} = -\partial_i\psi$$

$$\Gamma^i_{0j} = [H - \frac{\partial_i\psi}{1-2\psi}]\delta_{ij} = [H - a\psi']\delta_{ij}$$

$$\Gamma^i_{jk} = -2\delta^i_{(j}\partial_{k)}\phi + \delta_{jk}\delta^{il}\partial_l\phi \tag{2.9}$$

where all nonlinear terms(higher order terms in ψ,ϕ and their derivatives) have been ignored.

Curvature Tensor

The non-zero components of Riemann curvature tensor are:

$$R_{00} = -3H' + \nabla^2\psi + 3H(\phi' + \psi') + 3\phi''$$

$$R_{0i} = 2\partial_i\phi' + 2H\partial_i\psi$$

$$R_{ij} = [H' + 2H^2 - \phi'' = \nabla^2\phi - 2(H' + H^2)(\phi + \psi) - H\psi' - 5H\phi']\delta_{ij} + \partial_i\partial_j(\phi - \psi) \tag{2.10}$$

2.3 Perturbed Stress Energy Tensor

2.3.1 Stress Energy Tensor of a perfect fluid

Considering the homogeneous and isotropic universe to be equivalent to a perfect fluid with no anisotropic stresses, we can write its stress energy tensor as :

$$T^\mu_\nu = (\rho + p)U^\mu U_\nu - p\delta^\mu_\nu \tag{2.11}$$

where ρ is the mean density and p is the pressure of the fluid and both are functions of time alone owing to homogeneity and isotropy.

The perturbation in $T_{\mu\nu}$ will then be

$$\delta T^\mu_\nu = (\delta\rho + \delta p)U^\mu U_\nu + (\rho + p)(\delta U^\mu U_\nu + U^\mu \delta U_\nu) - \delta p\delta^\mu_\nu \tag{2.12}$$

In the frame of the fluid i.e; the comoving frame, we have the four velocity $U_\mu = (1, 0, 0, 0)$ As we have seen before, δT^0_0 is a scalar, δT^i_0 a vector and δT^i_j a tensor. We can therefore do SVT decomposition involving only scalar perturbations to write them as:

$$\delta T^0_0 = \delta\rho$$

$$\delta T_0^i = \nabla i \delta\sigma$$
$$\delta T_j^i = \delta p \delta_j^i \tag{2.13}$$

2.4 Perturbed Einsteins Equations

Einstein Tensor

The Einstein tensor can be calculated from the perturbed metric and Riemann tensor.

$$G_{00} = 3H^2 + 2\nabla^2\phi - 6H\phi'$$

$$G_{0i} = 2\partial_i(\phi' + H\psi)$$

$$G_{ij} = -2(H' + H^2)\delta_{ij} + [\nabla^2(\psi - \phi) + 2\phi'' + 2(2H' + H^2)(\phi + \psi) + 2H\psi' + 4H\phi']\delta_{ij} + \partial_i\partial_j(\phi - \psi) \tag{2.14}$$

Substituting in the Einstein's equations simplifies down to 3 final equations as follows:
Trace-free partG_{ij}:

$$\partial_{(i}\partial_{j)}(\phi - \psi) = 0$$
$$\implies \phi = \psi \tag{2.15}$$

G_{00} component:

$$\nabla^2\phi = 4\pi G a^2 \rho\delta + 3H(\phi' + \phi) \tag{2.16}$$

G_{0i} component:

$$\phi' + H\phi = -4\pi G a^2(\rho + p)v \tag{2.17}$$

trace part G_i^i :

$$\phi'' + 3H\phi' + (2H' + H^2)\phi = 4\pi G a^2 \delta P \tag{2.18}$$

Notice that the second equation is very similar to the Poisson equation for gravitational potential. The above three equations can be combined with the background equations to arrive at a single equation for the potential as:

$$\phi'' + 3H(1 + w)\phi' - w\nabla^2\phi + (2H' + (1 + 3w)H^2)\phi = (4\pi G a^2)\delta p^{NA} \tag{2.19}$$

where w is the adiabatic speed = p'/ρ' and δp^{NA} is the non-adiabatic pressure perturbation which can be assumed to be zero. Finally, the third term can be proved to be equal to zero from the background equations. We therefore have:

$$\phi'' + 3H(1 + w)\phi' - w\nabla^2\phi = 0 \tag{2.20}$$

Chapter 3

Evolution of Gravitational potential

3.1 Length scales in the problem

From the previous chapter, we have:

$$\phi'' + 3H(1+w)\phi' - w\nabla^2\phi = 0 \tag{3.1}$$

In fourier space, it can be written as

$$\phi'' + 3H(1+w)\phi' + wk^2\phi = 0 \tag{3.2}$$

where k is the fourier mode wavenumber, which is equal to $2\pi/\lambda$ where λ is the physical wavelength of the perturbation. Clearly, there are two length scales involved in the equation: $H^{-1}\ and\ k^{-1}$, both having dimensions of length. Approximations to the equation can be made depending on which of the two dominates. This can be understood with the help of Fig(3.1) below.

Hubble radius $\mathbf{d_H}$ in different epochs

$$H = \dot{a}/a$$

Radiation domination : $a \propto t^{(1/2)}$

$$\implies d_H \propto a^2 \tag{3.3}$$

Matter domination : $a \propto t^{(2/3)}$

$$\implies d_H \propto a^{3/2} \tag{3.4}$$

The wavelength that 'enters the horizon/Hubble radius' at matter-radiation equality is called λ_{eq} . The phrase 'enters the horizon' refers to physical wavelength becoming smaller

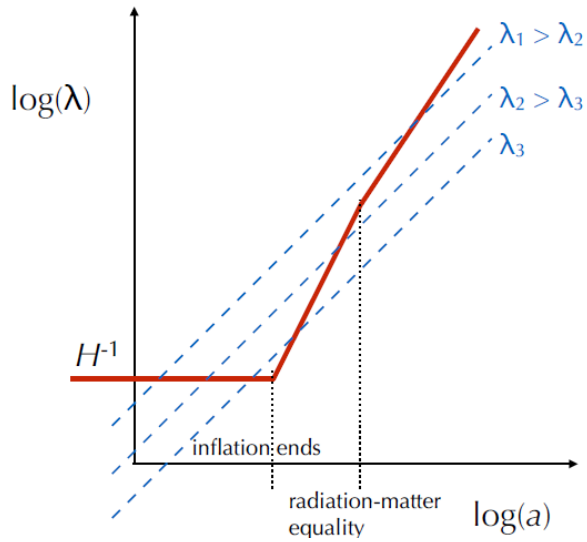

Figure 3.1: ln(λ)[$\lambda \propto a$], d_H vs ln(a) , Image Courtesy [3]

than Hubble radius. When the physical wavelength becomes smaller than the Horizon distance, there will be enough time for the interactions to occur in the age of the universe. In other words, the perturbations can be assumed to grow independent of the expansion of the universe. When $\lambda < d_H$ it is called Sub-Hubble regime and it's Super-Hubble regime when $\lambda > d_H$.

3.2 Curvature Perturbation tensor

Consider the quantity

$$R = \phi + \frac{2\rho}{3H}(\frac{\phi' + H\phi}{\rho + p}) \tag{3.5}$$

R is called the curvture perturbation tensor and it can be proved that R' vanishes in the limit of Super-Hubble scales.

Rearranging using background equations and equation of state $p = w\rho$, we have

$$R = -\frac{(5+3w)}{(3+3w)}\phi \tag{3.6}$$

Therefore, using the fact that R remains the same in the matter(w=0) and radiation dominated era(w=1), we have

$$\frac{5}{3}\phi_{MD} = \frac{3}{2}\phi_{RD}$$

$$\implies \phi_{MD} = \frac{9}{10}\phi_{RD} \tag{3.7}$$

for all $\lambda > d_H$

We can now study Eqn(3.2) in different eras under approximations.

3.3 Radiation dominated era

For radiation dominated era, we have w = 1/3 and H = 1/2t = $4/\tau^2$(conformal time) . Eqn(3.2) becomes:

$$\phi'' + \frac{4}{\tau}\phi' + \frac{k^3}{3}\phi = 0 \tag{3.8}$$

The solution to above equation is oscillatory and given by :

$$\phi \propto \frac{cos(k\tau/\sqrt{3})}{(k\tau)^2} \tag{3.9}$$

The complete solution can be found with the initial conditions set by inflation. In the limit of $k << H$ i.e; super-Hubble regime, the equation becomes:

$$\phi'' + \frac{4}{\tau}\phi' = 0 \tag{3.10}$$

The solutions are ϕ is constant and $\phi \propto a^{-5/2}$. We can ignore the decaying solution since the initial conditions are finite. Therefore, the super-horizon solution will be:

$$\phi \propto const. \tag{3.11}$$

3.4 Matter dominated era

For matter dominated era, w = 0 and therefore Eqn(3.2) is independent of k and we have the same solution for Sub and Super-Hubble regimes.

$$\phi'' + 3H\phi' = 0 \tag{3.12}$$

The solution would be either a constant or a decaying polynomial in time. We can discard the decaying solution since we do not want our initial conditions to blow up.

$$\phi \propto const. \tag{3.13}$$

Again, the const. is dependent on initial condition.

The evolution of gravitational potential can therefore be summarised as shown below:

Epoch	$K > K_{eq}$	$K < K_{eq}$
Radiation dominated	Sub horizon : $\Phi \propto a^{-2}\cos(a)$ Super horizon : $\Phi \propto$ constant	Super horizon : $\Phi \propto$ constant
Matter dominated	$\Phi \propto$ constant	$\Phi \propto$ const. = $9/10(\Phi_{rad})$

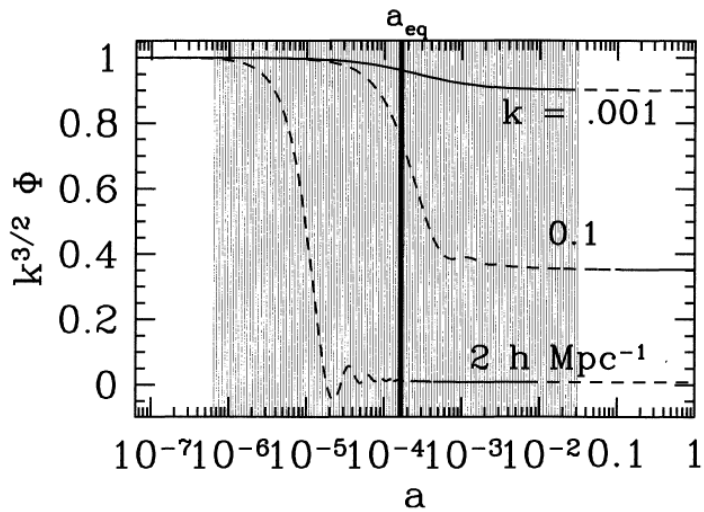

Figure 3.2: Evolution of potential in sub and super-Hubble regimes from radition to matter dominated era. Image Credit: Dodelson, Scott.Modern Cosmology

3.5 Transfer function

In order to relate the primordial potential to the potential today, we define Transfer function as :

$$\phi(k,a) = \phi_p(k)\, X\, (Transfer\ function(k))\, X\, (Growth\ factor(a)) \tag{3.14}$$

where the ϕ_p is the primordial potential generated during inflation, Growth factor describes the wavelength independent growth during late times. Fig(3.2) shows transfer function of different elements(cdm, baryon,photon) at a redshift of z =100. It can be noticed that all the components behave the same way in super-Hubble regime in all eras(as we know that ϕ is constant) while the sub-horizon evolution is different. Although not discussed in detail here, it can be seen how the transfer function of baryons trace that of dark matter. It can also been seen that for larger redshifts i.e; earlier epochs, the transfer function of baryons traces that of photons. This is because initially in radiation domination era, matter is coupled to radiation

until z=1000, after decoupling, the baryons fall into the gravitational potential wells created by dark matter.

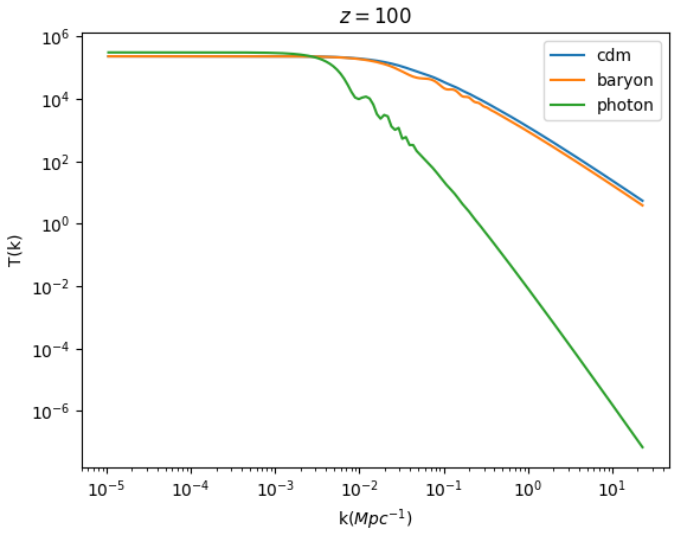

Figure 3.3: T(k) vs k : Transfer functions of different components at redshift z= 100. The plot was obtained using CAMB.

Chapter 4

Evolution of matter density perturbations

In this chapter, we'll study the evolution of density perturbations in the matter component which has been assumed to be pressure-less, the cold dark matter(cold because it is non-relativistic($mc^2 >> k_B T$)). In principle, we need to solve the complete Boltzmann equations with relevant collision terms to understand the evolution of density perturbations. It turns out that for the case of dark matter, the results can be derived from the relativistic fluid equations derived from the conservation of stress energy tensor.

4.1 Meszaros equation

From the conservation of stress energy tensor[4], we have the continuity fluid equation:

$$\delta' + (1 + \frac{p}{\rho})(\nabla \cdot v - 3\phi') + 3H(\frac{\delta p}{\delta \rho} - \frac{p}{\rho})\delta = 0 \tag{4.1}$$

and Euler equation :

$$v' + Hv - 3H\frac{p'}{\rho'}v = -\frac{\nabla \delta p}{\rho + p} - \nabla \phi \tag{4.2}$$

For dark matter, we may set $p = \delta p = 0$. Combining the equations 4.1 and 4.2, we get

$$\delta_m'' + H\delta_m' = \nabla^2 \phi \tag{4.3}$$

In eqn(4.3), the potential is considered to be sourced only by matter perturbations when averaged over time, since we have already seen that the potential in radiation era is oscillatory and vanishes when time averaged. From Eqn(2.15), we can write :

$$\delta_m'' + H\delta_m' - 4\pi G a^2 \rho_m \delta_m = 0 \tag{4.4}$$

Setting $y = a/a_{eq}$ and combining the above equation with background eqn(1.13), we get the **Meszaros equation**:

$$\frac{d^2\delta_m}{dy^2} + \frac{2+3y}{2y(1+y)}\frac{d\delta_m}{dy} - \frac{3}{2y(1+y)}\delta_m = 0 \tag{4.5}$$

The change of variable is to bring eqn(4.4) to a form that can be easily solved. The above eqn(4.5) is a second order differential equation which can be solved using Mathematica. The solutions are of the form:

$$\delta_m \propto 2 + 3y$$

$$\delta_m \propto (2+3y)ln(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}) - 6\sqrt{1+y} \tag{4.6}$$

Radiation dominated era:

In radiation dominated era, we have $y = a/a_{eq} << 1$ ant therefore the growing mode solution will be

$$\delta_m \propto ln(y) \propto ln(a) \tag{4.7}$$

Matter dominated era:

In matter dominated era, $y = a/a_{eq} >> 1$ and the growing mode solution is :

$$\delta_m \propto y \propto a \tag{4.8}$$

4.1.1 Super-Hubble evolution

From the first equation in Eqns(2.15), we have:

$$\delta = -\frac{2}{3}\frac{k^2}{H^2}\phi - \frac{2}{H}\phi' - 2\phi \tag{4.9}$$

For super Hubble evolution, we have $k << H$ and so the first two terms can be ignored. From Sec 3.3 and Sec 3.4, we have seen that the potential is constant in the super Hubble approximation. Therefore, we have $\delta_m = -2\phi = const.$ in the Super-Hubble regime. The evolution of matter density can be summarized in the figure (4.1). Length scales below k_{eq} are always in super-Hubble regime and therefore while length scales above k_{eq} are in sub-Hubble during radiation era and therefore grow logarithmically as seen.

As they become sub-Hubble during matter dominated era, they grow linearly with scale factor.

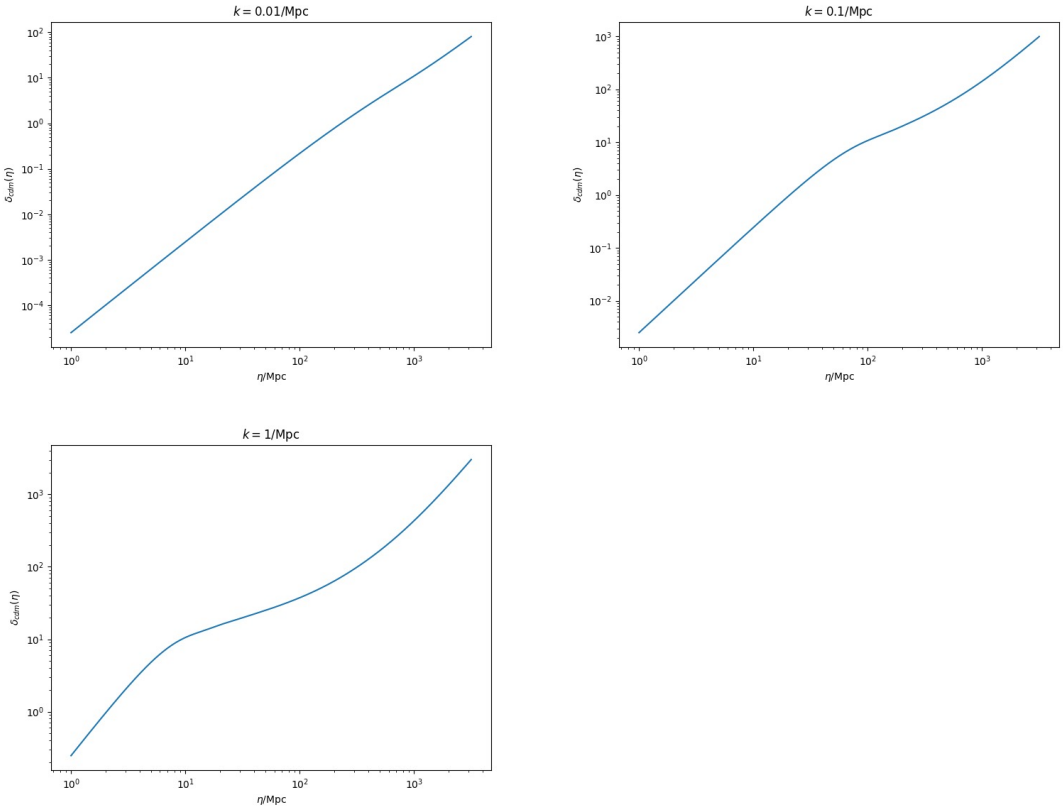

Figure 4.1: Evolution of dark matter density perturbations during different epochs for length scales = 0.01,0.1(k_{eq}),1Mpc^{-1}. All plots were obtained using CAMB.

4.2 Linear Matter Power Spectrum

4.2.1 Definition

The matter power spectrum P(k) is the square of amplitude of the fourier modes of the density perturbation. It is defined as

$$< \delta_m(\bar{k})\delta_m(\bar{k}') >= (2\pi)^3 P(k)\delta^3(\bar{k} - \bar{k}') \tag{4.10}$$

From Eqn.(4.3) and Eqn.(3.14)[5], the dark matter power spectrum can be derived to be :

$$P(k, a) = 2\pi^2 \delta_H^2 \frac{k^n}{H_0^{n+3}} T^2(k) (\frac{D(a)}{D(a = 1)})^2 \tag{4.11}$$

The matter power spectrum can be obtained from CAMB. The initial conditions were set for $n_s = 0.965$

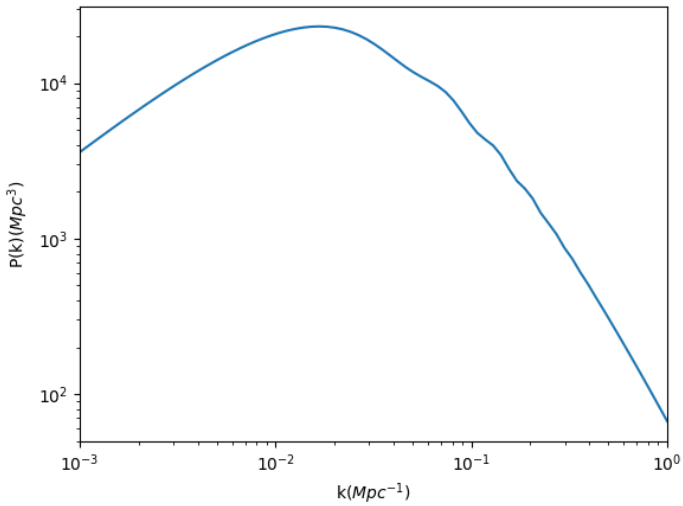

Figure 4.2: Linear matter power spectrum obtained using CAMB

4.2.2 Measured Matter Power Spectrum

The linear matter power spectrum can be measured from the two-point correlation functions of galaxy distribution. **Correlationfunction** The distribution of galaxies in the universe is not random. The two-point correlation function ζ measures the probability of finding a galaxy within a given distance from any galaxy and is given by :

$$P = (ndV)^2(1 + \xi_g) \tag{4.12}$$

where n is the average number density of galaxies in given volume dV. Similarly, we can also define the two point correlation function between two locations for total matter density in the universe as :

$$\begin{aligned} < \rho(x)\rho(y) >= (\bar{\rho})^2 < (1 + \delta(x))(1 + \delta(y)) > \\ = \rho^2(1 + < \delta(x)\delta(y) >) \\ = \rho^2(1 + \xi(|x - y|)) \end{aligned} \tag{4.13}$$

where $< \delta(x) >= 0$ as we are averaging over all space and ξ is a function of $|x - y|$ owing to the principle of homogeneity.

Power Spectrum :

The power spectrum is calculated as the 3D fourier transform function of the correlation function as :

$$P(k) = 2\pi \int_0^\infty dr\ r^2 \frac{sin(kr)}{kr} \xi(r) \tag{4.14}$$

Below is the linear dark matter power spectrum obtained from CAMB plotted along with the galaxy clustering power spectrum from SDSS BOSS DR1(2003) data.

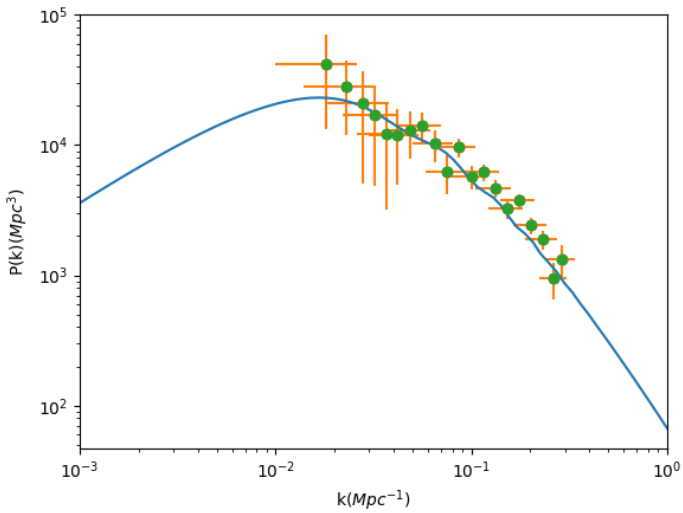

Figure 4.3: Linear matter power spectrum today (z=0) plotted against galaxy clustering power spectrum data from SDSS

4.2.3 Turn-around scale

The shape of power spectrum can be understood as follows. The larger k i.e; smaller wavelengths enter the Hubble radius early in radiation dominated era(Ref. Fig(3.1)). Therefore these density pertrubation modes evolve as ln(a) as derived in Sec(4.1). Compare this to the larger wavelength modes(or small k) which enter Hubble radius in matter dominated era and therefore have their density contrast $\propto$ a(τ). This implies that as the k increases(wavelength decreases), the growth rate becomes lesser($\propto ln(a)$)and therefore their

power is suppressed(since power spectrum is the amplitude of fluctuations). The characteristic scale at which turn-over occurs is set by λ_{eq} (which is understood from Fig(3.1)).
The linear matter power spectrum is therefore traced by the galaxy clustering power as observed today which implies that the baryonic matter sees the same potential as the dark matter at late times, after decoupling from radiation.

Chapter 5

Summary

The Cosmological Principle states that the universe is statistically homogeneous and isotropic on large scales. But on small length scales, about the order of distance between galaxies, we see that the universe is structured rather than being homogeneous.In order to study how structure formation occurred, we need to understand the linear cosmological perturbation theory. Introducing perturbations to the FLRW metric and stress-energy tensor of perfect fluid, we can solve the Einstein's equations upto first order in scalar perturbations. The resulting equations can be combined with background evolution equations to obtain a single equation for the Newtonian potential ϕ as (Eqn.3.1):

$$\phi'' + 3H(1+w)\phi' + k^2 w\phi = 0$$

The above equation an be solved under tow different approximations :$k >> H \ or \ k << H$ corresponding to sub-Hubble and super-Hubble regimes respectively. From the conservation of the curvature perturbation tensor R(Eqn3.5), it can be found that the potential is always a constant in the super-Hubble regime of a given era. The sub-hubble evolution of potential is dependent on what era we are looking at. In the radiation dominated era, the potential is suppressed and is oscillatory. In the matter dominated era, the potential is again found to be constant(which is 9/10 of the constant potential in radiation dominated era(Eqn3.7)). The evolution of matter perturbations can now be studied by making use of the Poisson equation and relativistic fluid equations. In the radiation dominated era, the matter density evolves logarithmically in sub-Hubble regime while it is a constant in the Super-Hubble regime. In the matter dominated era, the matter density evolves linearly with the scale factor while it remains constant in the super-Hubble regime. This behaviour of

matter density in sub-Hubble regime is expected as we know that in radiation dominated era, the growth of matter perturbations is diluted by photons while that is not the case during matter dominated era, leading to the linear growth of matter perturbations. Finally, we have the matter power spectrum which gives us the square of amplitude of density perturbations which can be computed from the transfer function(Eqn(4.11). The matter power spectrum can also be found by taking the fourier transform of two-point correlation function of galaxy clustering. This data has been obtained from SDSS(2003,[9]) and was plotted against the matter power spectrum derived from perturbation theory and the results are shown in Fig(4.3). The matter power spectrum is traced by the galaxy power spectrum and therefore it can be concluded that the baryonic distribution is similar to the dark matter distribution today(at late times i.e; long after decoupling). The current understanding is that baryons collapse in the potential wells of dark matter halos after decoupling, leading to the formation of stars, galaxies surrounded by the dark matter halo.

Bibliography

[1] See, http://www.sdss.org/science/orangepie/

[2] See, https://map.gsfc.nasa.gov/media/121238/index.html

[3] Image credit : Aseem Paranjape, IUCAA VSP 2017 lectures

[4] See, http://www.damtp.cam.ac.uk/user/db275/Cosmology/Lectures.pdf

[5] L. Sriramkumar. An introduction to inflation and cosmological perturbation theory, 2009; arXiv:0904.4584

[6] Dodelson, Scott. Modern Cosmology. Academic Press, 2003.

[7] Schneider, P. Extragalactic Astronomy and Cosmology: an Introduction. Springer, 2006. Bosch

[8] Mo, Houjun, et al. Galaxy Formation and Evolution. Cambridge University Press, 2010.

[9] M. Tegmark et al, Astrophys. J. 606, 702 (2004).