Generalized Kac-Moody Algebras from CHL Dyons

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Plan of talk

- Introduction: Learning to Count
- Denominator Formulae for Lie Algebras
- Denominator Formulae from by taking the square-root of Dyon degeneracies!
- Conclusion

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Motivation

- There has been tremendous progress in providing a microscopic counting of states that contribute to the entropy of supersymmetric black holes.
- More generally, one is interested in understanding the number of BPS states in different contexts such as chiral primaries in CFT's, black hole entropy.
- As the amount of supersymmetry is reduced, these numbers can change as we vary the vevs of various moduli. This change can be modeled like reactions in chemistry:

$$A + B \longleftrightarrow C \ .$$

  The numbers thus vary as one crosses *walls of marginal stability*.

Motivation

- The general idea is to ask whether we can combine all BPS states into a single multiplet ('module') of some algebra. Are walls of marginal stability related to walls of Weyl chambers?
- Clearly, this algebra should be such that its irreps are infinite dimensional. [Harvey-Moore]
- Charges of the BPS states then correspond to weight vectors of the algebra.
- Our motivation has been to make this idea more precise.
- The laboratory to test these ideas is given by $1/4$-BPS states in four-dimensional $\mathcal{N} = 4$ supersymmetric compactifications in string theory where there has been some progress in recent times. [Sen,DVV,David,Jatkar,Dabholkar]

Motivation

- The main idea of our work is to explore the relation between Generalized Kac-Moody (GKM) algebras and BPS states in string theory.
- It has been anticipated by Harvey and Moore that the algebra of BPS states in toroidal compactification of heterotic string is closely related to a GKM algebra.
- We pursue an interesting relation between the square root of modular form that counts BPS states with GKM algebras. In the process, we construct a family of ‘new’ GKM algebras.
- These GKM algebras play a role similar to the spectrum-generating algebras such as $so(4,2)$ for the Hydrogen spectrum or $su(3,1)$ for the isotropic 3D harmonic oscillator. [Mukunda, O’Raifeartaigh, Sudarshan(1965)]

Entropy of a black hole

- Bekenstein has argued that a black hole must carry entropy proportional to the area of its horizon

$$S_{BH} = \frac{1}{4} A_H \ .$$

- Using semi-classical arguments, Hawking showed that a black hole radiates like a blackbody at a temperature called the Hawking temperature, T_H.
- In Einstein-Maxwell theory, there exist spherically symmetric black holes that carry electric/magnetic charge, the so-called Reissner-Nordstrom black holes.
- These black holes have an inner and outer horizon that coincide in the *extremal* limit. Extremal black holes are stable and have $T_H = 0$.

Black Hole Entropy

- Can we provide a microscopic description of black hole entropy?

$$\frac{A_H}{4} \stackrel{?}{=} \log\left[\text{degeneracy of microstates}\right] \equiv S_{\text{stat}}$$

- While the answer is negative, in general, there exist situations where one does have a positive answer.
- The first example is due to Strominger and Vafa(1996). This is a 5D black hole solution in type II string theory compactified on $K3 \times S^1$.
- The microscopic counting was done (in perturbative string theory) using a system of D1-D5 D-branes, which, in the limit of large charges matched the macroscopic BH entropy.

Black Hole Entropy

- The success of Strominger and Vafa makes one to look for more realistic examples, say, in four-dimensions?
- In 1995, even before this result, Ashoke Sen noticed that for a family of electrically charged black holes in heterotic string compactified on T^6, the microscopic counting gave a non-zero answer while the corresponding blackhole had zero horizon area!
- Sen argued that higher derivative corrections in string theory would lead to a 'stretched horizon' with area of order of ℓ_S^2. He showed that, modulo a multiplicative constant, the microscopic entropy did match such a picture! Thus, one has

$$S_{\text{stat}} = S_{BH} + \text{higher derivative corrections}$$

- How does on compute the corrections to S_{BH}?

Higher derivative corrections to S_{BH}

- An important contribution to the macroscopic computation of entropy came from the work of Wald.
- He proposed a method of computing blackhole entropy in diffeomorphism invariant theories of gravity and not just Einstein gravity.
- This entropy, S_{BHW}, has been shown to reduce to S_{BH} for Einstein gravity.
- In 2005, Dabholkar revisited the Sen computation and showed that S_{BHW} exactly reproduced the microcanonical entropy after incorporating R^2 corrections.
- For large blackholes with non-zero area, the R^2 corrections are sub-leading to the BH entropy.

Counting microstates

- From a microscopic viewpoint, one thus wishes to count the number of ‘microstates’ corresponding to a given ‘macrostate’.
- The addition to supersymmetry to the story makes it possible to find weak coupling regimes where this counting can be carried out and ‘safely’ extrapolated to strong coupling regimes.
- BPS states(solutions) are special states(solutions) that preserve some fraction of supersymmetry.
- Typically, there exist indices analogous to the Witten index that count configurations corresponding to a specific macrostate.

Some counting problems

We list a few such examples that appear in the context of string theory/M-theory/field theory:

- Chiral primaries in four-dimensional superconformal field theories.
- Counting of (dual) giant gravitons in type IIB on $AdS_5 \times X^5$.
- The counting of instanton configurations in 4D $\mathcal{N} = 2$ supersymmetric gauge theories.
- The Gopakumar-Vafa Schwinger computation in M-theory on $CY3 \times S^1$ – this counts the D0-D2 configurations.
- Dyonic ($\frac{1}{4}$ BPS) states in 4D string theories with $\mathcal{N} = 4$ supersymmetry – related to Sen black hole. [Focus of this talk]

Organising the counting

- Using ideas from statistical mechanics, one sees that it is simpler to construct generating functions for the counting problems.

Organising the counting

- Recall that the canonical partition function in statistical mechanics is the weighted sum over configurations of a fixed energy with weight $\exp(-\beta E)$. For fixed E, the coefficient of $\exp(-\beta E)$ gives the number of states with energy E.
- The counting of BPS states is done in a similar manner. Introduce a fugacity (q_i) for every charge (n_i). One then defines

$$\mathcal{Z}(\mathbf{q}) = \sum_{\mathbf{n}\in L} d(\mathbf{n})\ \mathbf{q}^{\mathbf{n}}$$

  where $d(\mathbf{n})$ is the number of BPS states (microstates) in a macrostate with charge vector $\mathbf{n}$. The charge vector $\mathbf{n}$ is valued in a lattice, L, due to charge quantization.

Our example

- For the heteroic string theory on T^6, electrically charged states get mapped to the states of the heterotic string. This is given by (with $q = e^{2\pi i\tau}$ and $2n = \mathbf{q}_e^2$)

$$\mathcal{Z}(q) = \sum_n d(n)\; q^n = \frac{1}{\eta(\tau)^{24}}\;,$$

  where $\eta(\tau) = q^{\frac{1}{24}}\prod_{m>0}(1-q^m)$ is the Dedekind eta fn.
- These black holes preserve $\frac{1}{2}$ the supersymmetry.
- $\eta(\tau)^{24}$ is a modular form of $SL(2,\mathbb{Z})$ of weight 12.

$$\eta(-1/\tau)^{24} = (-\tau)^{12}\;\eta(\tau)^{24}\;.$$

  The behaviour of $d(n)$ at large n can be obtained using the modular property.

Our example

- Dyonic blackholes, ie., black holes carrying both electric and magnetic charge, preserve $\frac{1}{4}$ of the supersymmetry.
- No simple description of such states exist. However, Dijkgraaf, Verlinde and Verlinde proposed that their degeneracy formula is generated by a Siegel modular form of weight 10 under $Sp(2,\mathbf{Z})$: $\Phi_{10}(\mathbf{Z})$. [hep-th/9607026]
- $$\mathcal{Z} = \sum_{n,\ell,m} d(n,\ell,m)\; q^n r^\ell s^m = \frac{1}{\Phi_{10}(\mathbf{Z})}\ .$$

  where $q=\exp(2\pi i z_1)$, $r=\exp(2\pi i z_2)$, $s=\exp(2\pi i z_3)$ and $\mathbf{Z}=\left(\begin{smallmatrix} z_1 & z_2 \\ z_2 & z_3\end{smallmatrix}\right)\in \mathbb{H}_2$. Further $2n=\mathbf{q}_e^2$, $\ell=\mathbf{q}_e\cdot\mathbf{q}_m$, $2m=\mathbf{q}_m^2$.
- The formula is S-duality invariant i.e., the modular form is invariant under a $SL(2,\mathbb{Z})$ subgroup of $Sp(2,\mathbb{Z})$.
- It reproduces the entropy of (large) dyonic black holes.

Our example

- There exist a family of string theories, the CHL compactifications, with $\mathcal{N}=4$ supersymmetry that are obtained as $\mathbb{Z}_N$ orbifolds of the heterotic string on T^6 that we just considered.
- Jatkar and Sen constructed a family of genus-two modular forms, Φ_k, of weight k, that play a role analogous to Φ_{10}, for $N=2,3,5,7$ and $(k+2)=\frac{24}{N+1}$. [hep-th/0510147]
- The orbifolding breaks the S-duality group to the subgroup $\Gamma_1(N)$ of $SL(2,\mathbb{Z})$ and hence the modular group is a subgroup of $Sp(2,\mathbb{Z})$.
- David, Jatkar and Sen showed that these modular forms have a product representation similar (in spirit) to the one for the Dedekind eta function. [hep-th/0602254]

The Weyl Denominator Formula

- We will see that the square root of the modular forms $\Phi_k(\mathbf{Z})$ appear as the Weyl denominator formulae for a family of GKM algebras, $\mathcal{G}_N$.
- We will discuss this through a series of examples starting with the Lie algebra $su(3)$, then we discuss the affine Kac-Moody algebra before considering $\mathcal{G}_N$.
- Consider a representation $\mathcal{V}_\lambda$ with highest weight λ of some Lie algebra. The character of the representation is given by

$$\chi_\lambda(\mathbf{q}) = \sum_\mu d(\mu)\ \mathbf{q}^\mu\ ,$$

where μ is a weight vector and $d(\mu)$ its multiplicity. Note the formal similarity with our definition of $\mathcal{Z}$.

The Weyl Denominator Formula

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- For example, in the spin-j irrep (multiplet) of $su(2)$

$$\chi_j(q) = q^j + q^{j-1} + \cdots + q^{1-j} + q^{-j} = \frac{q^{j+1} - q^{-j-1}}{q - q^{-1}} \ .$$

  Here the weight is the J_3 eigenvalue.
- The character can be rewritten as

$$\chi_\lambda = \frac{\sum_{w\in W} (-1)^w \ \exp[w(\rho+\lambda)]}{\sum_{w\in W} (-1)^w \ \exp[w(\rho)]} \ .$$

$su(3)$: a toy example

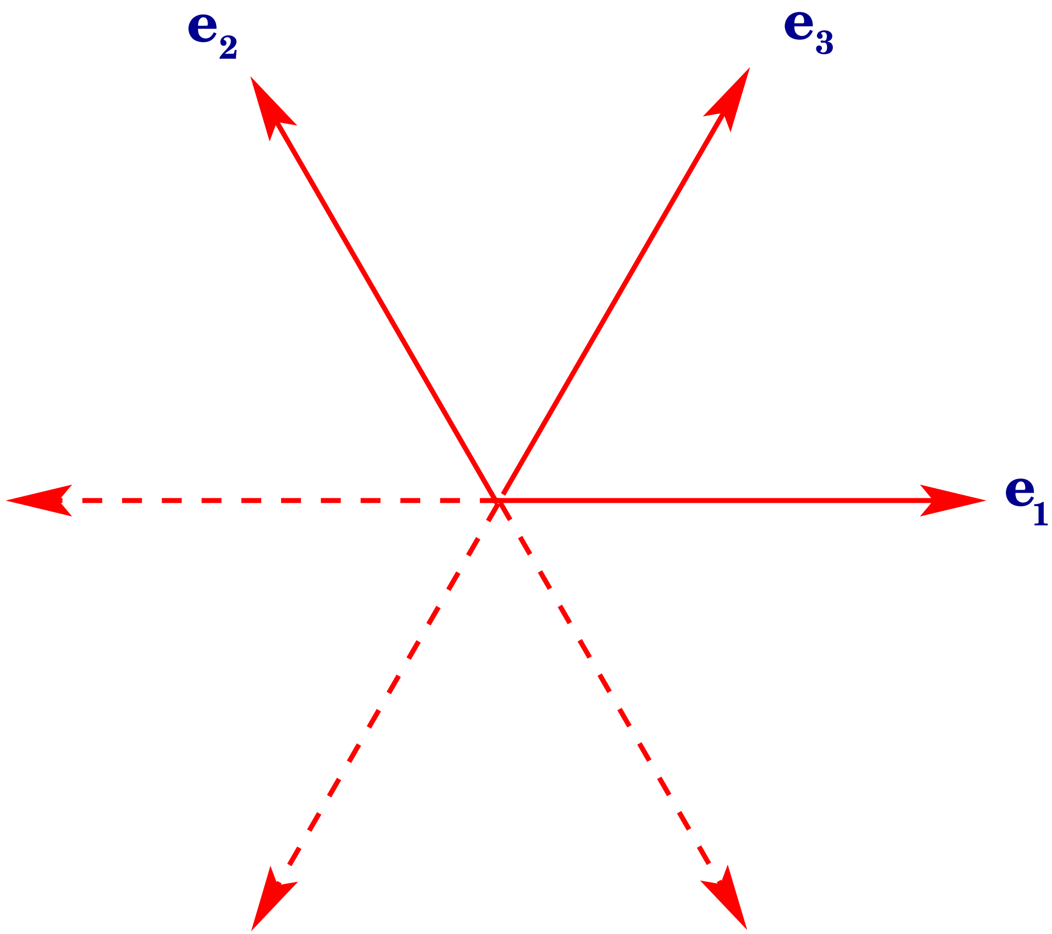
- The Lie algebra of $su(3)$ can be decomposed as

$$su(3) = L_+ \oplus H \oplus L_- \ .$$

where H is the Cartan sub-algebra, $L_+ = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are the positive roots ('raising operators'), and $L_- = (-\mathbf{e}_1, -\mathbf{e}_2, -\mathbf{e}_3) = -L_+$ are the negative roots('lowering operators').

- It is useful to picture the roots as (weight) vectors in a two-dimensional space as $su(3)$ is a rank-two Lie algebra.
- The Weyl group, $\mathcal{W}$, is generated by elementary reflections of the simple positive roots. It is isomorphic to S_3 the permutation group in three elements.
- The Weyl vector, ρ, is defined as the half the sum of all positive roots. One has $\rho = e_3$ for $su(3)$.

Weyl group and chamber for $su(3)$



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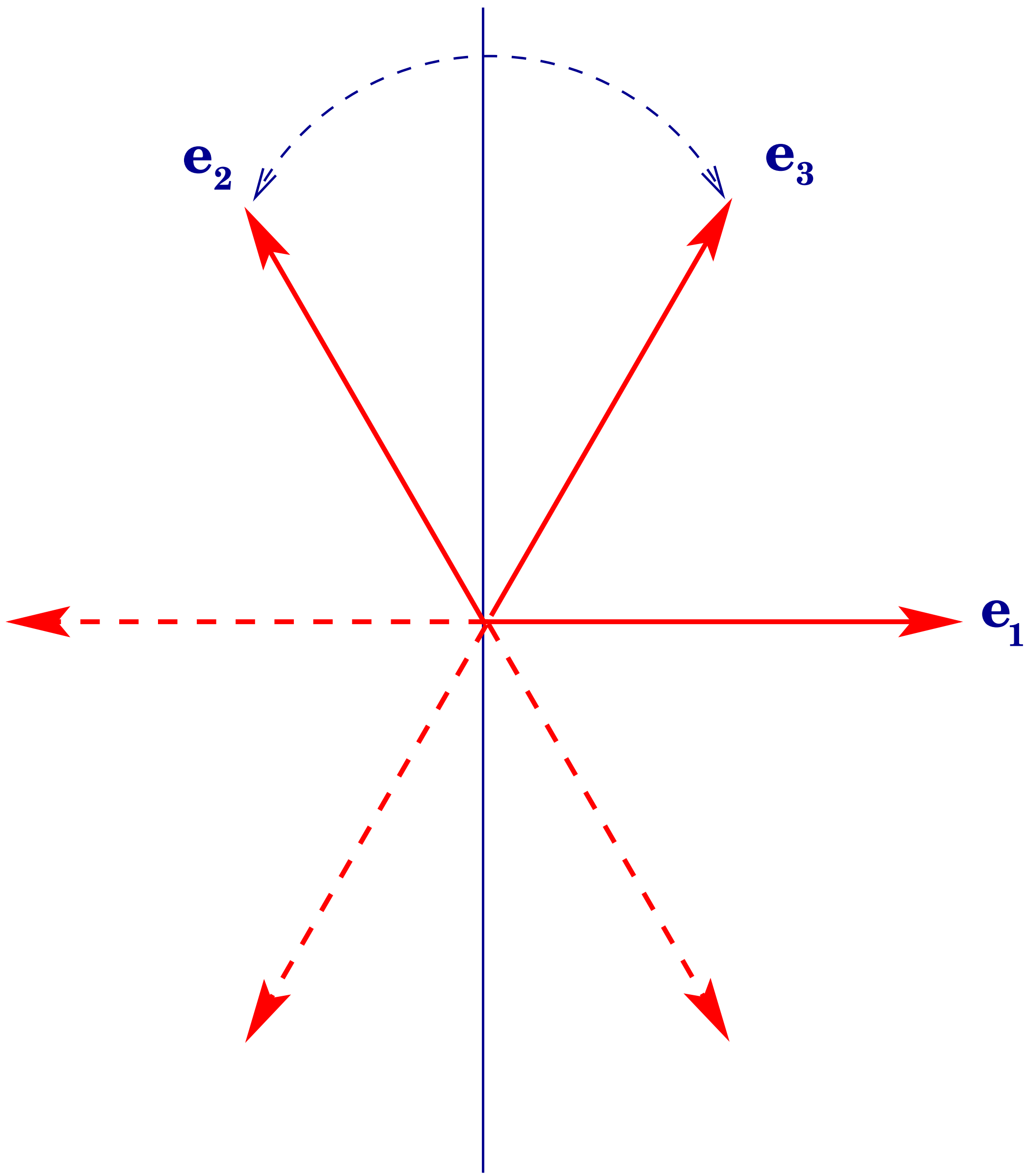

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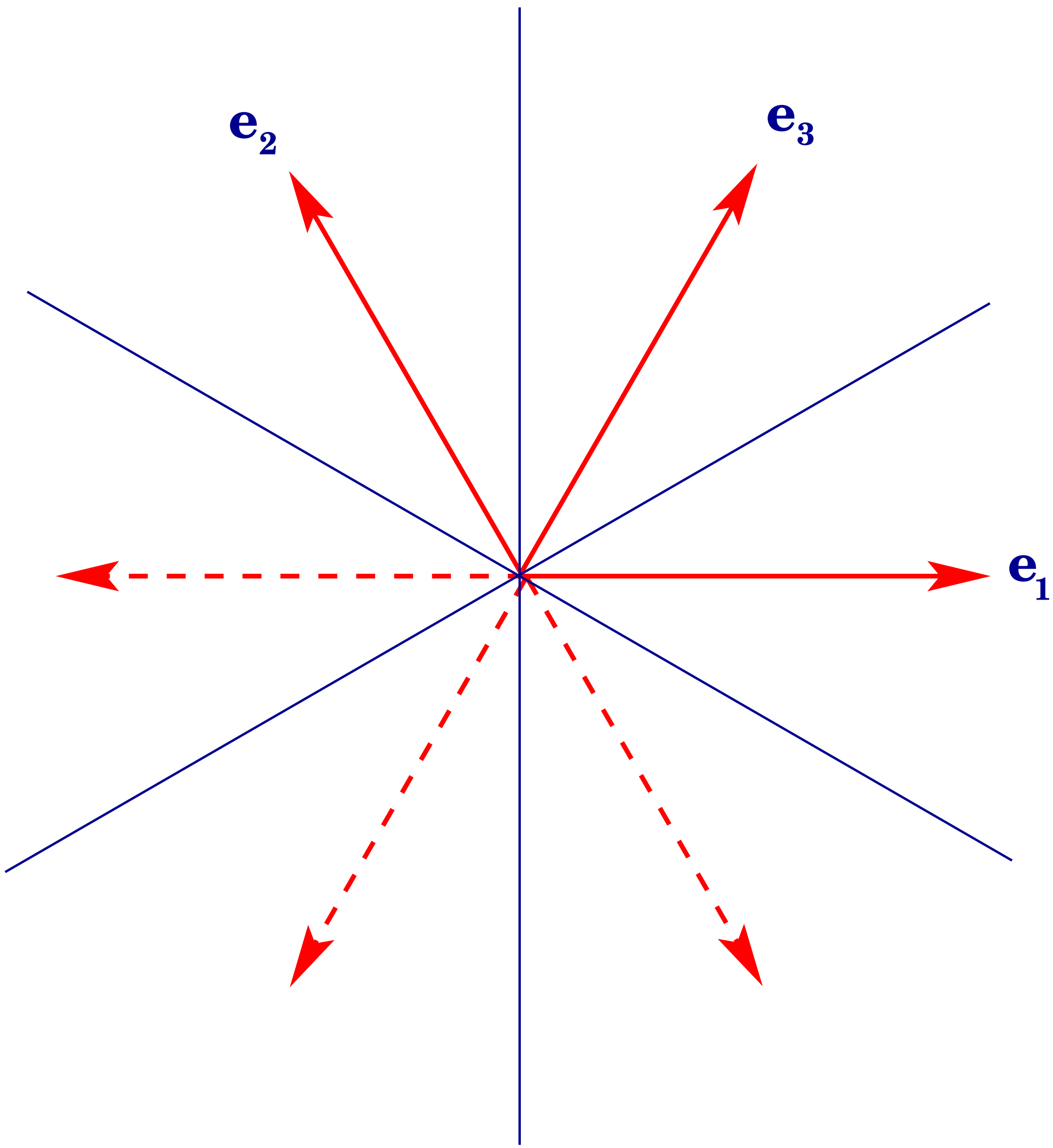

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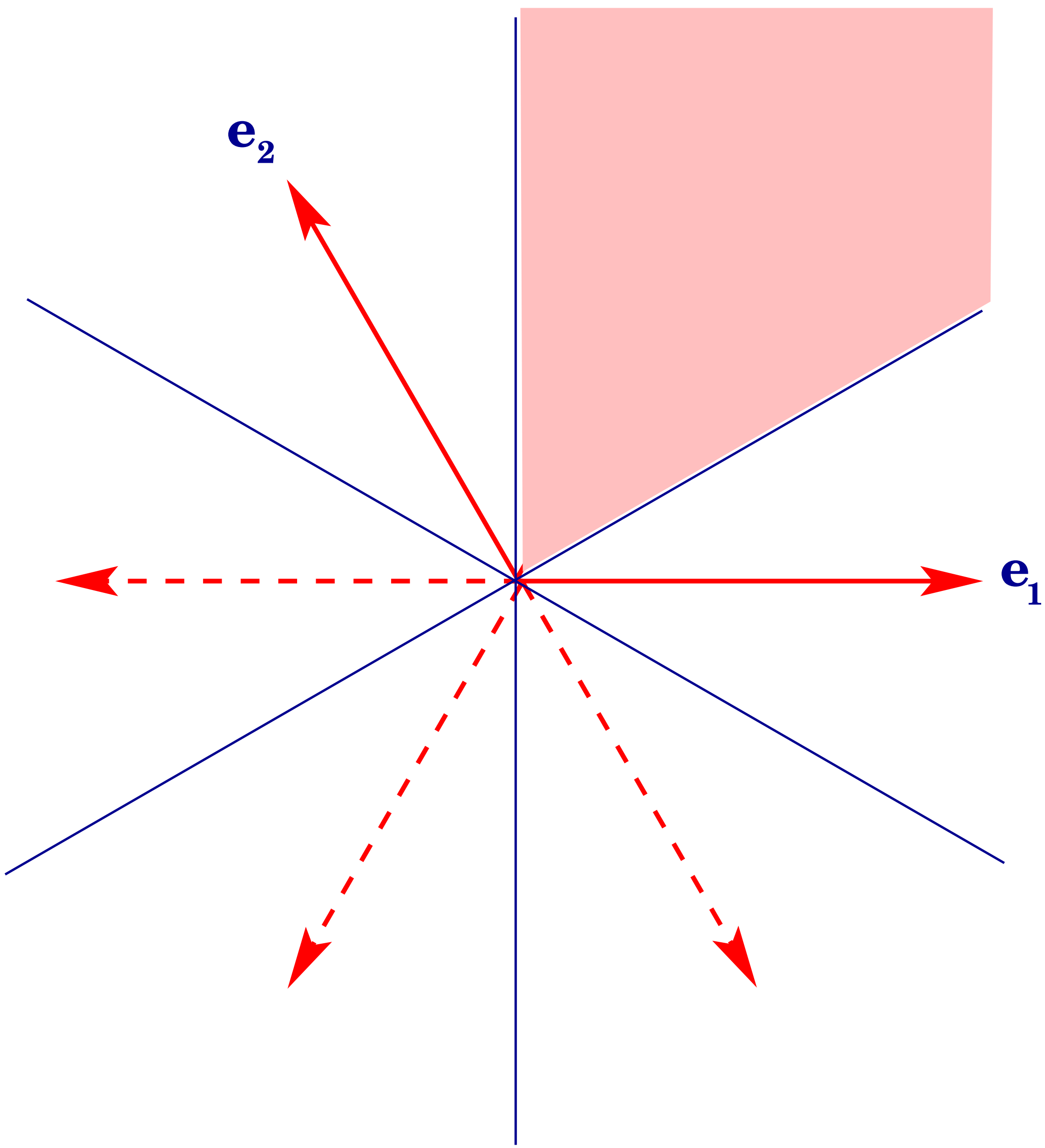

The Weyl denominator formula

- Recall the denominator of the character formula

$$\Sigma \equiv \sum_{w \in \mathcal{W}} (-1)^w \exp[w(\rho)]$$

- For $su(3)$, this reads

$$\Sigma = e^{\mathbf{e}_3} - e^{\mathbf{e}_2} + e^{-\mathbf{e}_1} - e^{-\mathbf{e}_3} + e^{-\mathbf{e}_2} - e^{\mathbf{e}_1}$$

Writing $x = e^{-\mathbf{e}_1}$ and $y = e^{-\mathbf{e}_2}$ and $xy = e^{-\mathbf{e}_3}$, we get

$$\Sigma = (xy)^{-1} - x^{-1} + x - xy + y - y^{-1} = \frac{(1-x)(1-y)(1-xy)}{xy}$$

Note that the appearance of a product running over positive/negative roots.

The Weyl denominator formula

- The appearance of the product is a general feature of (most) Lie algebras.
- Let
$$\Pi = \prod_{\alpha \in L_+} \Big(1 - \exp\left[-\alpha\right]\Big)^{\text{mult}(\alpha)} ,$$
where we allow for roots with multiplicity unlike $su(3)$ where all roots appear with multiplicity one.
- Re-defining Σ by multiplying by $e^{-\rho}$, we write
$$\Sigma = \sum_{w \in \mathcal{W}} (-1)^w \exp[w(\rho) - \rho]$$
- One then has the identity: $\Sigma = \Pi$

The affi ne denominator formula

- Several subtleties arise as the algebra is infinite dimensional.
- For the product side, defining the Weyl vector as the sum of all positive roots, needs regulation. However, $[w(\rho) - \rho]$ behaves well. So the sum part of the identity goes through.
- The product part doesn't quite work. This is due to the appearance of roots with zero norm. Recall that every root in the finite Lie algebra gives rise to an $su(2)$ sub-algebra. These imaginary roots give rise to a Heisenberg-Weyl Lie algebra (as in the harmonic oscillator).
- These roots need to be added to the L_+ with multiplicity and then the identity $\Sigma = \Pi$ holds. [MacDonald]

The simplest affi ne algebra: $\hat{su(2)}_1$

- This is an extension of the Lie algebra $su(2)$ with root α_1. Add to it a new root α_0 with length two and inner product -2 with α_1.
- One has

$$L_+ = \Big(n(\alpha_1 + \alpha_0), n\alpha_1 + (n-1)\alpha_0, (n-1)\alpha_1 + n\alpha_0 \Big| n \in \mathbb{Z}_+ \Big) .$$

- The identity $\Sigma = \Pi$ becomes the Jacobi triple identity [Weyl-Kac]

$$-i\vartheta_1(\tau, z) = q^{1/8} r^{-1/2} \prod_{n=1}^{\infty} (1 - q^n) \left(1 - q^{n-1} r\right) \left(1 - q^n r^{-1}\right)$$

$$= \sum_{n \in \mathbb{Z}} (-1)^n \, q^{\frac{(n-1/2)^2}{2}} \, r^{n-1/2} .$$

The denominatory identity for GKM algebras

- Borcherds generalised the identity for GKM algebras.

$$\prod_{\alpha \in L_+} (1 - e^{-\alpha})^{\text{mult}(\alpha)} = e^{-\rho} \sum_{w \in \mathcal{W}} (-1)^w \, w(e^{\rho} \sum_{\alpha \in L_+} \epsilon(\alpha) e^{\alpha}) \; ,$$

- The key modification appears in the Σ side of the story.
- There are imaginary simple roots, ie., those with zero or negative norm.
- Such terms lead to a correction to the summation side. In the absence of such roots, $\epsilon(\alpha)$ vanishes except for one term and we recover the earlier formula.
- $\epsilon(\alpha)$ is defined to be $(-1)^n$ if α is the sum of n pairwise independent, orthogonal imaginary simple roots, and 0 otherwise.

The GKM superalgebra $\mathcal{G}_1$

- The modular form $\Phi_{10}(\mathbf{Z})$ is a well-studied object mathematically. Gritsenko shows that its square-root, $\Delta_5(\mathbf{Z})$ has a sum representation using a method due to Maaß.
- Gritsenko and Nikulin showed that $\Delta_5(\mathbf{Z})$ appears as the denominator of a GKM algebra, $\mathcal{G}_1$ and provided a product representation for it as a consequence.
- The algebra $\mathcal{G}_1$ is a rank-three hyperbolic Kac-Moody algebra. Hyperbolic Kac-Moody algebras have been well classified and they exist only a finite number with rank between 3 and 10 and none at higher rank. At rank two, there are infinite hyperbolic Kac-Moody algebras.
- $\mathcal{G}_1$ has a Weyl vector, a nice presentation for its Weyl group.

The GKM superalgebras $\mathcal{G}_N$

- Following Gritsenko and Nikulin, we defined $\Delta_k(\mathbf{Z})$ via
$$\Phi_k(\mathbf{Z}) = \left[\Delta_{k/2}(\mathbf{Z})\right]^2 .$$
- The product representations given by David, Jatkar and Sen enabled us to read out the positive roots along with their multiplicities.
- Negative multiplicities do occur – they correspond to fermionic roots and hence it is a Lie superalgebra.
- The sum side enabled us to obtain the multiplicities of imaginary roots.
- This works only for $N = 2, 3, 5$. For $N = 7$, we got fractional multiplicities and hence the GKM algebra interpretation doesn't work.

Properties of the GKM algebras: $\mathcal{G}_N$

- The GKM algebras, $\mathcal{G}_N$, constructed for the modular forms Δ_k for $(k+2) = \frac{24}{N+1}$, and $N = 2, 3, 5$ are found to have identical real roots to the $\mathcal{G}_1$ GKM algebra.
- The Weyl group and Weyl vector do not change upon orbifolding.
- For a root of the form ta_0 where a_0 is a primitive imaginary root with zero norm the multiplicity is generated by a generating function of the form

$$1 - \sum_{t \in \mathbb{N}} m(ta_0)\; q^n = \prod_{n \in \mathbb{N}} (1-q^n)^{\frac{k-4}{2}} (1-q^{Nn})^{\frac{k+2}{2}}$$

This correctly reproduces the multiplicity obtained by Gritsenko and Nikulin for $\mathcal{G}_1$.

Conclusions

- The precise connection between the GKM algebras and the algebra of BPS states has not yet been addressed. We would like to understand the sense in which the various roots, real and imaginary, are the building blocks of $1/4$-BPS states.
- A deeper understanding of the representation theory of the GKM algebras, and their relation to the BPS states will shed valuable light in the understanding of the whole idea, and help extend it to other string theories.
- The implication of the appearance of imaginary simple roots for the the dyon spectrum is not very clear to us.
- Cheng and Verlinde have recently shown that Weyl transformations of the GKM algebra are related to wall-crossing formulae.

Conclusions and future directions

- Since the modular form Φ_k is the square of $\Delta_{k/2}$, we anticipate that there are two copies of the GKM algebra $\mathcal{G}_N$ that naturally appear. Can we provide a physical interpretation for this?
- The Gopakumar-Vafa counting of BPS states in M-theory is closely related to counting of 'colored' plane partitions. It is known that there is a product representation for these objects. Is there an algebraic interpretation for this?
- Recent work of Moore and Gaiotto on wall-crossing may have bearing on our pursuit in relating walls of marginal stability to walls of Weyl chambers.
- It would be of interest to extend our considerations to $\mathcal{N} = 2$ string theories.

Thank you