Indian Institute of Technology Madras
Quantum Mechanics for Engineers. PH305.
Assignment 2. 01/09/2004.

1. Consider the set of (in general complex) functions of a real variable x such that $\int_a^b |\phi(x)|^2 \, dx < \infty$ (i.e. does not diverge), for some real numbers a and b. Let ϕ and ψ be two representative functions of this set. Addition of members of this set are defined as $(\phi + \psi)x = \phi(x) + \psi(x)$. A linear vector space structure is endowed to these functions when we treat the functions as representations of these vectors, say the position representation. Thus we "take apart" the function to write: $\langle x|\phi\rangle = \phi(x)$. (0) Verify that this is a Hilbert space if the inner product is defined as:
$$\langle\phi|\psi\rangle = \int_a^b \phi^*(x)\psi(x)dx.$$
This Hilbert space is denoted as $L^2[a, b]$.

2. Identify the functions that belong to $L^2(-\infty, \infty)$:
$$(a)\ (x^2 + 1)^{-1/4} \quad (b)\ e^{-x}\cos(x) \quad (c)\ e^{-1/x^2}$$
$$(d)\ (\sin(x))/x \quad (e)\ x^3 e^{-x^2} \quad (f)(\tanh(x))/x$$

3. Consider the operator A which is such that:
$$\langle x|A|\phi\rangle = \frac{d}{dx}\phi(x).$$
This is the "derivative" operator. Show that it is *not* Hermitian on $L^2[-\infty, \infty]$. Remember that an operator A is Hermitian if for any pair of vectors $|\phi\rangle$ and $|\psi\rangle$, $\langle\phi|A|\psi\rangle = \langle\psi|A|\phi\rangle^*$. In fact prove that A is *anti*-Hermitian, that is $A^\dagger = -A$. (Hint: When in doubt, integrate by parts. Also the functions have to vanish at infinity faster than $1/sqrtx$ if they have to be square-integrable.) It is customary to be sloppy and write $A = d/dx$, when in fact it is a particular representation.

4. In $L^2(-\infty, \infty)$ find the adjoints of the following operators:

$$(a)\ \frac{d}{dx}\,x \quad (b)\ \frac{d^2}{dx^2} + \omega^2 \quad (c)\ x\,\frac{d}{dx} \quad (d)\ x\,\frac{d}{dx}\,x$$
$$(e)\ x^2\,\frac{d^2}{dx^2} \quad (f)\ \exp(ia\,\frac{d}{dx}).$$

where ω and a are real constants.

5. Show that while the expectation value of any Hermitian operator for any state is real, it is purely imaginary for anti-Hermitian operators. Show that id/dx is Hermitian on $L^2[-\infty, \infty]$.

6. Is $x^2 \frac{d}{dx}$ a linear operator?

7. Define the exponential of an operator A as $\exp(A) = I + A + A^2/2! + \cdots$. If A is linear, if $\exp(A)$ linear?

8. Prove that $e^{a\frac{d}{dx}}\phi(x) = \phi(x+a)$. Hence show that $e^{iP/\hbar}\phi(x) = \phi(x+a)$, where P is the Hermitian operator $P = -i\hbar d/dx$. Thus P is said to "generate translations". This is identified as the **momentum operator** in quantum mechanics. Show that the generalization to 3-dimensional space is $P = -i\hbar\nabla$.

9. Show that the functions $1, \sqrt{2}\sin(2\pi kx), \sqrt{2}\cos(2\pi kx), (k = 1, 2, 3, \cdots)$ form an orthonormal basis for $L^2[0, 1]$.

10. (a) Find the eigenfunctions and eigenvalues of the operator P as defined above on the space $L^2[0, 1]$. (b) Consider the subspace of $L^2[0, 1]$ consisting of functions such that $\phi(0) = \phi(1)\exp(2\pi i\alpha)$, where α is a real number between zero and one-half. Is this subspace itself an Hilbert space? Find the spectrum of the momentum operator restricted to this space. (c) For the subspace consisting of functions that vanish at the boundaries: $\phi(0) = \phi(1) = 0$, find the spectrum of the momentum operator.

11. If A and B are two operators, the commutator is defined as $[A, B] = AB - BA$ and the anti-commutator is defined as $\{A, B\} = AB + BA$.

If A and B are Hermitian, find if the following are Hermitian, anti-Hermitian, or neither. (a) $A+B$, (b) AB, (c) BA, (d) $\{A,B\}$, (e) $[A,B]$, (f) $i[A,B]$. Show that $\mathrm{tr}([A,B])=0$.

12. Define the position operator as that operator for which $|x\rangle$ are the eigenkets with eigenvalues x. So that $\langle x|X|\phi\rangle = x\phi(x)$. The position operator in the position representation is a simple multiplication operation. Show that the commutator $[X,P]=i\hbar$ Hence find the trace of the commutator between X and P. Note that this contradicts the last part of the previous problem. X and P are operators in an **infinite** dimensional Hilbert space and their commutator trace need not vanish. This shows that there cannot be any finite dimensional representations of the operators X and P that obey the above "canonical" commutation relation.

13. If two operators A and B are such that $[A,B]=0$ they are said to **commute**. Prove that two commuting Hermitian operators share the same set of eigenvectors.

14. On the Hilbert space $L^2[0,1]$ find the momentum representation of the momentum operator. Find the momentum representation of the position operator.

15. An operator U is said to be **unitary** if $U^\dagger U = UU^\dagger = I$. (a) Show that the eigenvalues of an unitary operator all lie on the unit circle, i.e, if λ is an eigenvalue then $|\lambda|=1$. (b) Prove that the eigenvectors form an orthonormal set. (c) if A is Hermitian prove that e^{iA} is unitary. (d) Show that unitarity implies that the inner product is preserved between two vectors if both are acted on by the same unitary operator.

16. An operator A is called **anti-unitary** if for any pair of vectors $|\phi\rangle$ and $|\psi\rangle$, $\langle A\phi|A\psi\rangle = \langle\phi|\psi\rangle^*$. where $A|\phi\rangle \equiv |A\phi\rangle$ etc. Note that we have been careful not to define its adjoint. Prove that it is an "anti-linear" operator as $A(\alpha|\phi\rangle) = \alpha^*|\phi\rangle$. Can it be represented by a matrix? Later we will encounter an important operator the "time-reversal" operator which is anti-unitary.

17. Show that for any pair of vectors $|\phi\rangle$ and $|\psi\rangle$ the following inequality

(**Cauchy-Schwarz inequality**) holds:

$$|\langle\phi|\psi\rangle|^2 \leq \langle\phi|\phi\rangle\langle\psi|\psi\rangle.$$

Equality holds iff the vectors are linearly dependent. In order to prove the inequality consider the norm of the vector $|\psi\rangle+\alpha|\phi\rangle$, and choose the complex number α suitably. Using this inequality prove the "triangle" inequality $||\phi+\psi|| \leq ||\phi||+||\psi||$, where the norm is defined by $||\phi||^2 = \langle\phi|\phi\rangle$.

18. Show that the set of all complex $n \times n$ matrices form a linear vector space. Find the dimensionality of this space. This space becomes an inner product space if the inner product of two matrices A and B are defined by $brA|B\rangle = \mathrm{Tr}(A^\dagger B)$, where $A^\dagger$ is the Hermitian conjugate of A. Show that
(a) If A is Hermitian then

$$\mathrm{Tr}(A^2) \geq \frac{1}{n}(\mathrm{Tr}A)^2.$$

(b) If A is an arbitrary $n \times n$ matrix and U is an unitary matrix (also $n \times n$) then

$$\mathrm{Tr}(A^\dagger A) \geq \frac{1}{n}|\mathrm{Tr}(UA)|^2.$$

19. Write down the most general possible 2×2 Hermitian matrix. Find its eigenvalues and verify that these are real.