Indian Institute of Technology Madras
Quantum Mechanics for Engineers. PH305.
Assignment 3. 30/09/2004.

1. Let $\hat{q}$ and $\hat{p}$ be the quantum position and momentum operators. Evaluate the commutator:
$$\left[\hat{q},\, \exp\left(\frac{i\hat{p}a}{\hbar}\right)\right],$$
and hence prove that
$$\exp\left(\frac{i\hat{p}a}{\hbar}\right)|q\rangle = |q+a\rangle.$$

2. If F and G are functions that can be expressed as power series, prove that
$$[\hat{q},\, G(\hat{p})] = i\hbar\frac{\partial G}{\partial p}, \quad [\hat{p},\, F(\hat{q})] = -i\hbar\frac{\partial F}{\partial q}.$$

3. An electron is subject to a uniform, time-independent magnetic field of strength B in the positive z direction. Ar $t = 0$ the electron is kown to be in an eigenstate of $\mathbf{S}\cdot\hat{\mathbf{n}}$ with eigenvalue $\hbar/2$, where $\hat{\mathbf{n}}$ is an unit vector lying in the xz plane, that makes an angle β with the z axis. (a) Obtain the probability of finding the electron in the $s_x = \hbar/2$ state as a function of time. (b) Find the expectation value of S_x as a function of time. (c) Show that the answers make sense in the extreme cases of $\beta = 0$ and $\beta = \pi/2$.

4. Consider a partile in one dimension whose Hamiltonian is given by
$$H = \frac{p^2}{2m} + V(x)$$
By calculating $\left[\left[\hat{H}, \hat{x}\right], \hat{x}\right]$ prove that
$$\sum_{a'} |\langle a''|\hat{x}|a'\rangle|^2 (E_{a'} - E_{a''}) = \frac{\hbar^2}{2m},$$
where $a'\rangle$ is an energy eigenket with eigenvalue $E_{a''}$.

5. Consider a particle in three dimensions whose Hamiltonian is given by

$$H = \frac{\mathbf{p^2}}{2m} + V(\mathbf{x})$$

By calculating $[\hat{\mathbf{x}} \cdot \hat{\mathbf{p}}, H]$ obtain

$$\frac{d}{dt}\langle \hat{\mathbf{x}} \cdot \hat{\mathbf{p}} \rangle = \langle \hat{\mathbf{p}}^{\mathbf{2}}/m \rangle - \langle \hat{\mathbf{x}} \cdot \nabla V \rangle.$$

To identify this relation as the quantum analogue of the classical virial theorem, it is essential that the left-hand side vanishes. Under what condition would this happen?

6. Consider a free particle in one dimension whose initial state is

$$\langle q|\psi(0)\rangle = \frac{1}{\pi^{1/4}\sqrt{d}} \exp\left(ikq - q^2/2d\right)$$

Find the expectation and dispersion values and the postion and momentum operators at this initial state. Sketch the probability distributions in both the position and momentum. Prove that at a later time t:

$$|\langle q|\psi(t)\rangle|^2 = \frac{1}{\sqrt{\pi}d(t)} \exp\left(-(q - k\hbar t)^2/d^2(t)\right),$$

where

$$d(t) = d\sqrt{1 + \hbar^2 t^2/d^4}$$

What is the probability distribution in momentum at this time? Think about the corresponding classial situation to gain more insight.

7. A particle is trapped in an infinite one dimensional well of width a. if the initial state is $|\psi(0)\rangle = (|\phi_0\rangle + |\phi_2\rangle)/\sqrt{2}$, where $|\phi_0\rangle$ and $|\phi_1\rangle$ are the normalized ground and first excited states of this well. Find the probability that the state will remain unchanged after a time t, that is find the "autocorrelation" $|\langle\psi(0)|\psi(t)\rangle|^2$. At what time does this reach a maximum and what is this maximum value? Prove that for *any* initial state there exists a time that is independent of this state such that the initial state returns exactly. How does this time compare with what you found in the first part?

   (This forms a subject of current research called "revivals" and more information and articles can be found in the technical review article at: http://in.arxiv.org/abs/quant-ph/0401011)