Hamiltonian Chaos: Classical and Quantum Aspects

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Notes for the SERC school on Nonlinear Dynamics at PRL. Dec. 2004.

Lecture 1

1.1 Hamiltonian Systems: Preliminaries

Hamiltonian systems are a very special class of dynamical systems that are of central concern to physics. They encompass the classical mechanics of Newton and are the starting point for quantum mechanics.

The dynamics is completely specified by specifying the values of a minimum number of coordinates of the particles involved and their conjugate momenta. This minimum number, the *number of degrees of freedom*, or DoF, may require some thought to figure out.

Degrees of Freedom:

1. *The number of degrees of freedom of a planar pendulum is* **one**, *the angle it makes with the vertical say is enough to specify its position.*
2. *If we suspend the planar pendulum on a spring then the length is not a constant and therefore this system has* **two** *DoF.*
3. *A rigid rotator with a ball-and-socket joint has ?? DoF. If the rotator is no more rigid but is actually a spring. How many DoF?*
4. *How many DoF in a water molecule?*
5. *How many degrees of freedom does your arm have?! Without physically harming your neighbours, extend your arms out and check on first the number of DoF for your shoulders, then elbow joint, and then the wrists. According to one source the total is seven. Do you agree?*
6. *Speculative: Number of DoF of a swan's neck.*

Let the number of DoF be d. Let us state at the very outset that this is a very critical number. All that we understand reasonably well today are utmost 2 DoF systems! Detailed structures and behaviour of say even 3 DoF systems is beyond the scope of modern science. The key reason for this surprising inability is "chaos", which is potentially present for any system whose $d > 1$. It therefore remains a deep mystery as to how we have anything at all to say about complex systems such as liquids and gases.

Let the coordinates that make up the DoF be labelled $(q_1, q_2, \ldots, q_d)$. Then the *phase-space* is the space that also includes the conjugate momenta $(p_1, p_2, \ldots, p_d)$. Thus the phase-space is has $2d$ dimensions. This is the proper space in which Hamiltonian mechanics is to be studied. Every physical system is assumed to be described by an Hamiltonian H which is a scalar function on the phase-space and completely determines the future and past of the $2d$ variables. The central equations are the Hamilton equations of motion:

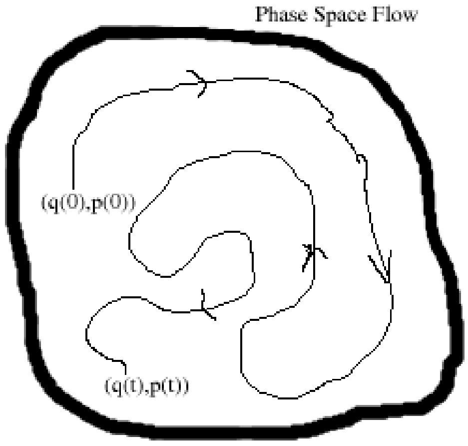

Figure 1.1: A phase space doodle

$$\boxed{\begin{aligned} \frac{dq_i}{dt} &= \frac{\partial H}{\partial p_i} \\ \frac{dp_i}{dt} &= -\frac{\partial H}{\partial q_i} \qquad (1.1) \end{aligned}}$$

These $2d$ first order ordinary differential equations are determined by the *single* phase-space function H. Note the $-$ sign in the momentum equation. Given any initial point $(q(0), p(0))$ it evolves to $(q(t), p(t))$ after a time t under the Hamiltonian equations of motion. Here $q(0)$ is written for the d position variable collectively, and similarly for momentum. Thus the Hamitonian generates a **flow** in the phase-space and classical mechanics is all about studying this flow. The Hamiltonian H could in general depend on all the $2d$ phase space variables as well as time t. If it depends on time t, the system is said to be non-autonomous, otherwise it is autonomous. For autonomous Hamiltonian systems, **trajectories in phase-space do not cross**. This follows from uniqueness of solutions of the set of first order o.d.e.s in Eq. (1.1).

If the trajectories crossed, then the partial derivatives of the Hamiltonian would have to be multi-valued at a given phase-space point. In fact they can be multivalued at different times if the system is non-autonomous and therefore phase-space trajectories can cross in this case. For such systems an extended $2d+1$ space which is phase-space plus time space is such that trajectories don't cross here. It is easy to see that the no-crossing condition places severe restriction for a flow in a plane, i.e. for $d = 1$ cases. Refer to the phase-space doodle in Fig. (1.1) . It is easy to see that for spaces where the phase-space is higher than 2 there is a possibility for fairly complicated trajectories. In fact all $d = 1$ Hamiltonian systems are trivially *integrable*, $H = constant$ will give the phase space trajectories. There is no chaos.

The moment we step out of $d = 1$ we see interesting physics, in particular deterministic "randomness" or "chaos" appears as a generic (meaning usual) phenomena. For instance take the class of Hamiltonians $H(q, p, t)$, with $d = 1$, that is nonautonomous one degree of freedom systems, the extended phase-space is 3D, (q,p,t). Sometimes these systems are referred to as "one and a half" degree of freedom systems.

As somebody said "generalities are for wimps, real scientists study examples", lets study examples.

Elementary Examples:

1. *1-d Harmonic oscillator:* $H = p^2/2 + \omega^2 q^2/2$. *The Hamiltonian equations of motion are linear. The phase space trajectories are nested ellipses. There is one equilibrium point at the origin. If any of these things are not clear, clarify before proceeding.*

2. *The simple pendulum Hamiltonian is* $H = p^2/2 - gl\cos(q)$. *Notice that the equations are now nonlinear. There are stable equilibrium points at the origin and for every* $(q = 2\pi k, p = 0)$, *phase space point, there are unstable, hyperbolic points at* $(q = \pi(2k+1), p = 0)$, *where* k *is any integer.*

3. **Exercise:** *Draw the phase-space trajectories of the so-called Harper Hamiltonian* $H = \cos(2\pi p) + g\cos(2\pi q)$ *for some constant* g. *In particular analyze the cases as* g *crosses unity.*

1.2 Welcome to 1.5 degrees of freedom

1.2.1 Examples

All this is autonomous so far. Lets step out of $d = 1$ into $d = 1.5$, that is nonautonomous 1 DoF systems. We will take our elementary examples and modify them slightly. We will parametrically force them, i.e. change their parameters, like the restoring force, the length of the pendulum, gravity etc., in time. The harmonic oscillator case is well-known, the solutions are the *Mathieu* functions, the oscillator can be stable or unstable depending on the forcing amplitude and frequency. The case of the pendulum and the Harper Hamiltonian leads to chaos. These are *nonlinear* systems, and it may be expected that almost all nonlinear systems that are parametrically forced, even if they have only one-degree of freedom, will be chaotic. Surprisingly very little material is found on parametrically forced pendulum and it has been the subject of a Ph.D. thesis very recently!

Consider the general $d = 1.5$ Hamiltonian, for some functions f and h

$$H(q,p,t) = f(p) \;+\; g(t)\,V(q) \tag{1.2}$$

where $g(t)$ is some smooth function of time t. In the case when $g(t)$ is a periodic function, of period T say, important simplifications occurs. As an illustration consider the cases when the kinetic energy term is of the usual form, i.e. $f(p) = p^2/2$ and the case $g(t) = g(1 + \epsilon\sin(2\pi t))$, for the two potential, the simple harmonic oscillator ($V(q) = q^2/2$) and the planar pendulum ($V(q) = \sin(q(t))$).

In Fig. (1.2) we see what happens to such parametrically forced oscillators. Firstly, due to the nonautonomous nature of the systems, the phase-space trajectories cross. However they seem to be forming much more regular patterns in the case of the harmonic oscillator, but the pendulum behaviour starts looking complicated for increasing forcing. Before going any further let us illustrate with the pendulum an important feature: *sensitivity to initial conditions.* Shown in Fig. (1.3) are now the fate of *two* initial conditions that are separated initially by a small amount. We see that the parametrically forced pendulum appears unstable, in the sense that these two trajectories quickly depart from each other, while the harmonic oscillator looks stable.

There is at best a statistical resemblance between the two initially nearby trajectories in the case of the pendulum, while the eye can discern no difference in the case of the harmonic oscillator. The harmonic oscillator can also be driven to instability, but the instability in the pendulum is of a very different nature. The instability that may arise in the harmonic oscillator is like the instability of a hill-top, anything dropped at the top runs away to infinity, while the pendulum instability seems to

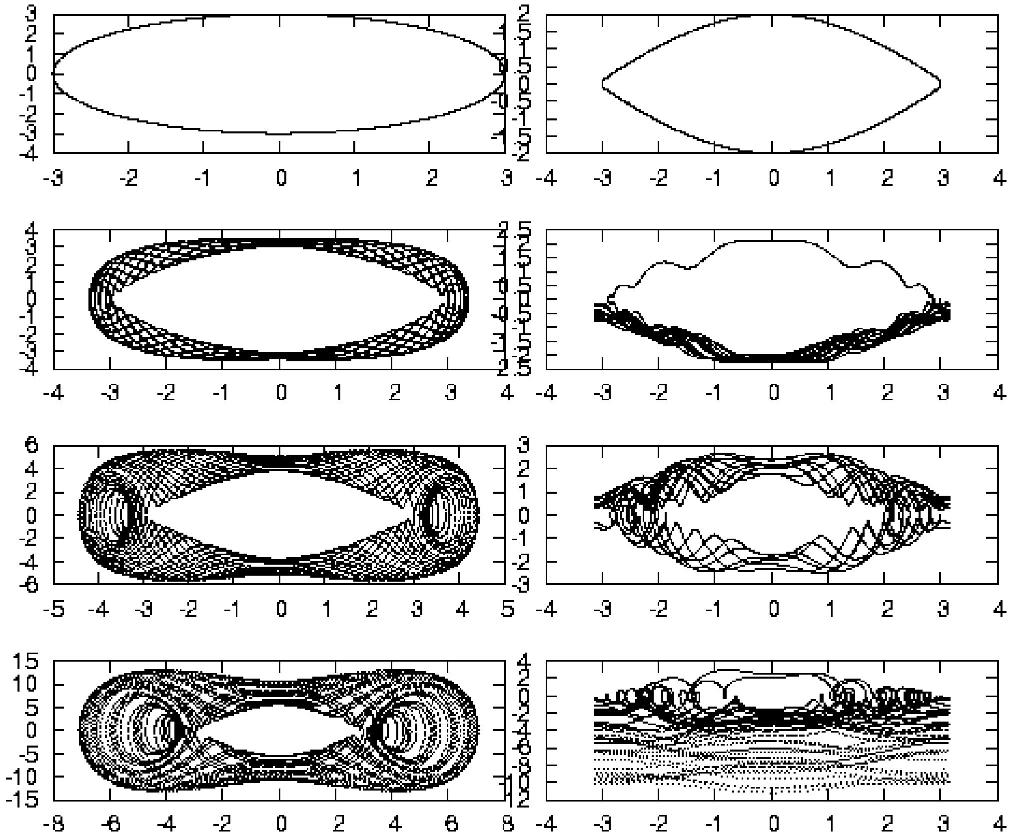

Figure 1.2: One phase-space trajectory of the parametrically forced harmonic oscillator (left) and the pendulum (right). Plotted is (q, p) as a function of time, all plots having the same initial condition $(q(0) = 3.0, p(0) = 0)$, and running for a time=100 (arb. units). The forcing function is $g(1+\epsilon\sin(2\pi t))$, and the parameters are $g = 1.0$, $\epsilon = 0, 2, 5, 10$ from top to bottom.

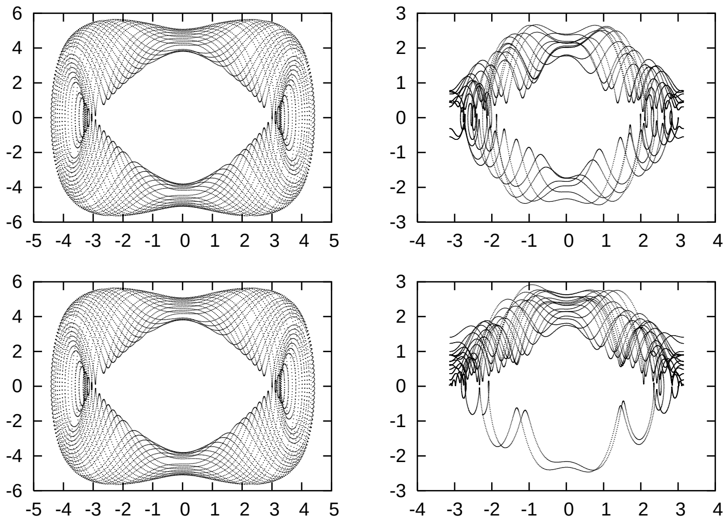

Figure 1.3: Tale of two initial conditions. On the left is the harmonic oscillator and the right is the pendulum. The initial conditions were $(q(0) = 3, p(0) = 0)$ (top) and $(q(0) = 3.0, p(0) = 0.001)$ (bottom). In both cases $\epsilon = 5$ and $g = 1$. The forcing function is same as that in Fig (1.2).

be there in *bounded* space. Even in the case when unboundedness arise, there is possibility of random looking recurrences in a very complicated manner. Thus this illustrates one of the central features of chaos in Hamiltonian systems:*instability coupled with confinement in finite regions of phase-space*. It should be remarked that there is Hamiltonian chaos in scattering systems as well, although we will not go into this aspect here.

While the pictures above are instructive, they are not terribly so. We have inferred that the harmonic oscillator is allegedly simpler, the pendulum is more complicated and perhaps sensitive to initial conditions. The pictures above have a disadvantage: this is only one trajectory, if we went ahead and plotted say 100 of them, so as to get a global understanding of phase-space behaviour, we will get some picture that looks worse than the Tamil Nadu Assembly in an excited state.

1.2.2 Stroboscopic Map

This is where we are helped by the fact that the forcing is periodic. We will choose to monitor the phase-space only at times separated by a period of the forcing. This will give us a discrete set of points, and the dynamics is specified by specifying where one point maps into the next time, the **stroboscopic map**. Although the equations of motion are invariant under a time translation by T, the solutions are in general not periodic, let alone periodic with period T (see the pendulum pictures! Also *more generally the system's symmetries need not be shared by the solutions*). Let $f_{t_0}^t$ map evolve phase-space points at time t_0 by time t, i.e.

$$f_{t_0}^t(x(t_0), p(t_0)) \ = \ (x(t_0 + t), p(t_0 + t)) \tag{1.3}$$

$f_{t_0}^t$ does not have have the continuous semigroup structure seen for autonomous systems, in the sense that $f_{t_0}^t \circ f_{t_0}^t \neq f_{t_0}^{2t}$. We cannot use the same map to propagate further by the same time t, as the map will depend on the base time, so that it is true that $f_{t_0+t}^t \circ f_{t_0}^t = f_{t_0}^{2t}$. However due to the time-periodicity of the Hamiltonian, the map over a time T will possess a discrete simple Abelian group structure.

$$f_{t_0}^T \circ f_{t_0}^T = f_{t_0}^{2T} \tag{1.4}$$

The map $f_{t_0}^T$ propagates phase-space at time t_0 by T and repeated applications n times gives us $(x(t_0+nT), p(t_0+nT))$. Also clearly we have $f_{t_0}^T = f_{t_0+T}^T$, so that we have to only consider stroboscopic maps with base points between any two periods, say $[0, T]$. There is still an *infinity* of such stroboscopic mappings, however all of these are equivalent in a strong mathematical sense, they are *topologically conjugate* to each other. This means that the maps produce qualitatively identical structures, if there is a periodic orbit in one, there will be a corresponding periodic orbit in the other. This is a generalization of what we study in a first course (and for many of us, last) of classical mechanics as a property of homogeneous potentials.

The stroboscopic $2 - dim$ map has *all* the information present in the extended phase-space, in a faithful manner. If there are periodic orbits in the phase-space, it will show up as a finite periodic collection of points on the map. In Fig. (1.4) we see how the map looks for the forced harmonic oscillator, and the forced pendulum.

Of Floquet and the Harmonic Oscillator.

The simplicity of the map for the parametrically forced harmonic oscillator is due to the fact that whatever may be the type of forcing function, $g(t)$, the stroboscopic map is a **linear** *one. This follows from because if $(x(t), p(t))$ is a solution then so is $x(t+T), p(t+T))$, but from the linearity of the differential equation the latter solution cannot be new, but must be a linear combination of the former. Thus we must have:*

$$\begin{aligned} x(n+1) &= a(t_0, T)x(n) + b(t_0, T)p(n) \\ p(n+1) &= c(t_0, T)x(n) + d(t_0, T)p(n), \end{aligned} \tag{1.5}$$

where $x(n) \equiv x(t_0 + nT)$ etc. This is a autonomous linear map, in the sense that the coefficients (a, b, c, d) don't depend on the iterate n, but they can depend on the base point t_0, and the period T. Also a very general property of Hamiltonian systems is reflected in the area-preserving condition that $ad - bc = 1$. Try to prove all these statements!

Floquet theory of linear differential equations with periodic coefficients, which predates a similar theorem due to Bloch which is applied for spatially periodic potentials in the Schrödinger equation and determines electronic band structure in solids, can be applied here, and the the Floquet exponents are the eigenvalues of the matrix $\{(a, b), (c, d)\}$. If the eigenvalues are complex and lie on the unit circle, the system is stable and gives rise to elliptic orbits as illustrated above. Otherwise they will appear as pairs of real numbers whose product is unity and in this case the orbits will generically be hyperbolas.

The exact expression (a, b, c, d) depend on the forcing function $g(t)$ and are in general quite hard to find. For instance if $g(t)$ is a cosine function, the solutions are not elementray functions, but special ones, the Mathieu functions. However there are important cases where we can write down analytically the stroboscopic map. As a fairly extensive warm up the exercise below recommended.

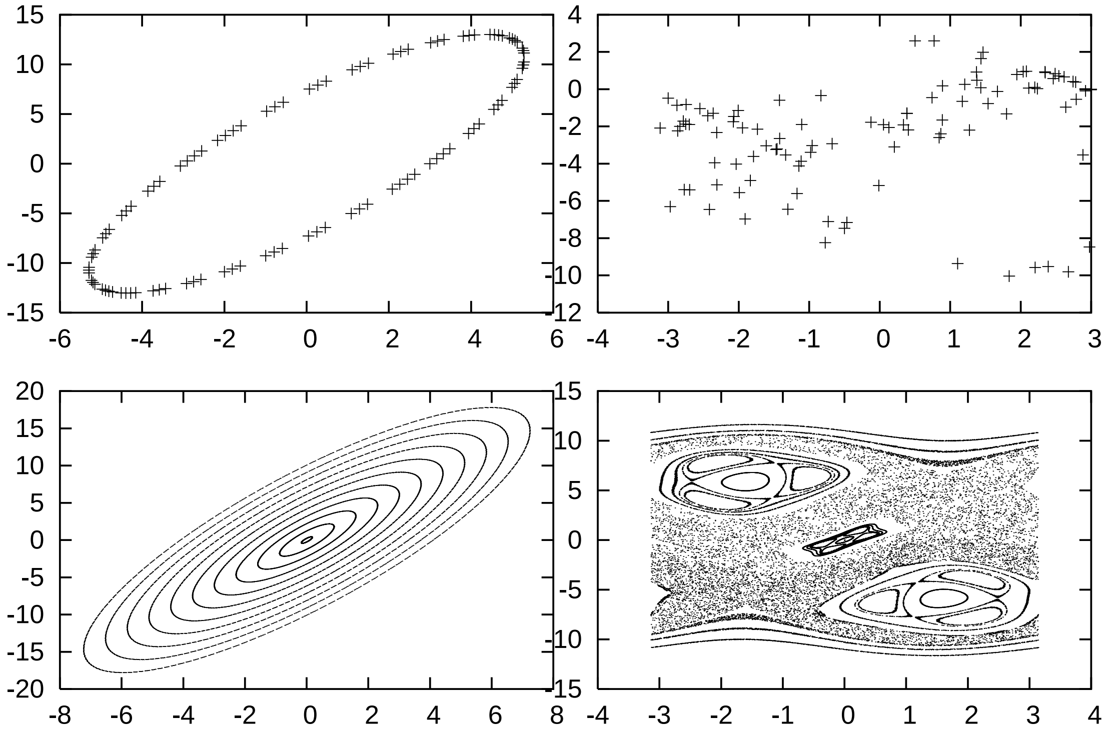

Figure 1.4: On the left is the harmonic oscillator and the right is the pendulum, stroboscopic maps. The top panel is for one hundred iterations with the initial conditions were $(q(0) = 3, p(0) = 0)$, while the bottom one features about 10000 iterations and many more initial conditions. In both cases $\epsilon = 10$ and $g = 1$. The forcing function is same as that in Fig. (1.2)

Problem. *Consider the Hamiltonian:*

$$H = p^2/2 + s_T(t)\omega^2 q^2/2, \tag{1.6}$$

where $s_T(t)$ is the periodic function:

$$s_T(t) = \begin{cases} 1 & 0 < t \leq T/2 \\ -1 & T/2 < t \leq T \end{cases} . \quad s_T(t) = s_T(t+T).$$

Find the linear stroboscopic map. Analyze it for stability. Note that we are periodically switching from a stable harmonic oscillator to an inverted "oscillator" and back again. Thus we may expect that overall stability may be crucially dependent on the switching time T. Verify that the stroboscopic map is area-preserving.

The simplicity of the forced harmonic oscillator is lost in the case of the forced pendulum in many ways. Firstly, as is clear from the figures we are talking of qualitatively different behaviour. The simple elliptic orbits are replaced by a mixture of random splatter of points and regular looking curves. These are intermixed in an intricate manner that is not readily visible to our eye. This is in fact the generic phase-space of $d = 2$ (or $d = 1.5$) Hamiltonian systems: there is a mixture of chaos and regularity, what is called (naturally) a "mixed phase-space". Complete chaos, and complete regularity are exceptional. There is a rigorous mathematical statement to this effect.

But apart from the complexity of the phase-space, even writing down the stroboscopic maps are difficult in general. This is because we need to be able to integrate the equations of motion over one time period. For example if you attempt the corresponding boxed exercise for the pendulum, you will learn a lot about Elliptic functions, and yet the map will not be simple enough. The map will be *nonlinear* in general. However we can in fact write down nonlinear stroboscopic maps for one kind of forcing function, which is essentially the derivative of the step function s_T, namely a train of *delta* functions. This amounts to impulses at regular intervals of time. This case is often referred to as the "kicked" Hamiltonians and form a very important and well-studied genre of low-dimensional Hamiltonian chaos problems.