Lecture 2

2.1 Kicked Hamiltonian Systems: Justforkix.

Consider Hamiltonians of the form

$$H = f(p) + \sum_{n=-\infty}^{\infty} \delta(t/T - n)\, V(q). \tag{2.1}$$

The Dirac delta function train is denoted in short as $\delta_T(t)$. The Hamiltonian equations of motion are:

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} = f'(p) \tag{2.2}$$
$$\frac{dp}{dt} = -\frac{\partial H}{\partial q} = -\delta_T(t) V'(q) \tag{2.3}$$

We need to now integrate these equations over one period. Let us choose as $t_0 = 0^+$, so that we are going to consider the stroboscopic map just *after* each kick. The variables just after the n^{th} kick (q_n^+, p_n^+) are to be evolved to the next kicked. Just after the kick, there is no force till just before the next kick. Therefore we can write:

$$p_{n+1}^- = p_n^+, \quad q_{n+1}^- = q_n^+ + T f'(p_n^+) \tag{2.4}$$

We have to now integrate over the kick number $n+1$ to arrive at q_{n+1}^+, p_{n+1}^+. Assuming that $f(p)$ is some smooth function of p, which does not itself have delta function dependencies, integrating the first of the equations of motion over the infinitesimal interval about the $n+1$ kick implies that the position of the particle is frozen. Integrating the second (force) equation gives us the impulse that is imparted, thus we can write:

$$q_{n+1}^+ = q_{n+1}^-, \quad p_{n+1}^+ = p_n^- - TV(q_{n+1}^+). \tag{2.5}$$

Finally combining the two sets of equations we get the stroboscopic map as:

$$\begin{aligned} q_{n+1} &= q_n + + T f'(p_n) \\ p_{n+1} &= p_n - TV'(q_{n+1}). \end{aligned} \tag{2.6}$$

We have dropped the $+$ superscript, as now every variable is defined as just after the respective kicks. Note that the second equation has the force $(-V'(q))$ evaluated at the *new* time. This follows from the equations of motion and make the above class of maps area-preserving for any functions f and V.

Exercises.

(a) Prove that map given in Eq. (??) is area-preserving. That is find the Jacobian of the transformation:

$$J = \begin{pmatrix} \partial q_{n+1}/\partial q_n & \partial q_{n+1}/\partial p_n \\ \partial p_{n+1}/\partial q_n & \partial p_{n+1}/\partial p_n \end{pmatrix}, \tag{2.7}$$

and show that $Det(J) = 1$.

(b) Find the stroboscopic maps in the case where the variables are monitored from just before the kicks, and in the case where they are monitored midway between kicks.

The stroboscopic maps in the family Eq. (**??**) present a very rich source of dynamical systems and it is quite accurate to say that we only partially understand them. Also while we have derived them from 1.5 dimensional systems, that is one-degree of freedom nonautonomous systems, it is important to note that they share the same phenomenology as the *Poincaré surface of section (PSOS) for two-degree of freedom autonomous Hamiltonian systems.* The PSOS of such systems are also area-preserving transformation of the same kind. PSOS are obtained by restricting ourselves to a given energy shell $H = E$ and observing the orbits as they cross a given section, traditionally as $x = 0$ in the positive $(p_x > 0)$ sense. The only difference between the PSOS and the stroboscopic maps is that the time of return to the PSOS is in general not the same and itself is a very complex, and ill-understood, function of phase-space. The popularity of maps such as in Eq. (**??**) is the simplicity of numerical work: we only have to iterate now and in principle there are no errors introduced due to truncation in integration schemes. Also certainly more analytical work is feasible in this approach.

2.1.1 Important Area-Preserving Maps in 2D

We now point out some of the important cases:

Linear Map. This case is when $V(q) = \omega^2 q^2/2$ and $f(p) = p^2/2$. We have already seen how *any* forced harmonic oscillator leads to linear maps, the kicked harmonic oscillator is no exception, but now the stroboscopic map is absolutely straightforward to study:

$$\begin{aligned} q_{n+1} &= q_n + T\,p_n \\ p_{n+1} &= p_n - T\,\omega^2 q_{n+1} = p_n(1-\omega^2 T^2) - T\omega^2\, q_n. \end{aligned} \tag{2.8}$$

The trace of the Jacobian, $\mathrm{tr}(J) = 2 - T^2\omega^2$, is an important quantity in this case and depending on whether $|\mathrm{tr}(J)| <$ or > 2 we have either stability (ellipses) or instability (hyperbolae). The phase space is the plane.

The Standard Map. This is the case of the kicked pendulum, $f(p) = p^2/2$ and $V(q) = -g\cos(2\pi q)/(2\pi)^2$. The stroboscopic map becomes:

$$\begin{aligned} q_{n+1} &= q_n + p_n \;(\mathrm{mod} 1) \\ p_{n+1} &= p_n - (K/2\pi)\,\sin(2\pi q_{n+1}). \end{aligned} \tag{2.9}$$

Here we have rescaled momentum: $T\,p \to p$, and $K = T^2\,g$ is a rescaled effective parameter. This is the only parameter the kicked pendulum depends on. This map is one of the most widely studied of simple chaotic systems, called variously as the "standard map", the "Chirikov map" or the "Chirikov-Taylor" map. The modulo operation is necessary as the coordinate is cyclic, an angle, that has been scaled to lie in $[0, 1)$. Thus the phase space is a cylinder, cyclic along q and infinitely extended along the p or the angular momentum direction.

THE HARPER MAP. This is the case of the kicked Harper Hamiltonian, $f(p) = \cos(2\pi p)/(2\pi)$ and $V(q) = g\cos(2\pi q)/(2\pi)$. The stroboscopic map becomes:

$$\begin{aligned} q_{n+1} &= q_n \; - \; T\sin(2\pi p_n) \; (\mathrm{mod}\; 1) \\ p_{n+1} &= p_n \; + g\,T\; \sin(2\pi q_{n+1}) \; (\mathrm{mod}\; 1). \end{aligned} \tag{2.10}$$

Thus this is a two parameter map (T and $g\,T$), defined on a *torus* which is a square which has its opposite sides identified, the topology is that of a south-Indian "medu vada", the only popular type of vada north of the Vindhyas. This map also has been used in many studies.

As you may imagine, choosing any $f(p)$ and $V(q)$ and possibly imposing different boundary conditions, such as above, will lead to a plethora of maps. In fact it is quite a challenge finding any that are *not* chaotic in some parameter range. Try it! There do exist such systems, and for the sake of recording a very rare nonlinear but nonchaotic map we state the following system.

AN INTEGRABLE MAP.

$$\begin{aligned} q_{n+1} &= q_n \; + \; p_n \\ p_{n+1} &= p_n \; + \; q_{n+1}(A - 2q_{n+1}^2)/(B + q_{n+1}^2). \end{aligned} \tag{2.11}$$

Here A and $B > 0$ are real constants. You can find the corresponding kicking potential easily and it represents an exact discretization of the quartic double-well potential. The integrability of such maps is intimately connected to very special properties of Elliptic functions. But such maps are very rare and even less understood. It is not clear how to check whether a given map is *not* chaotic, i.e. integrable. This forms an active area of research in Mathematical Physics.

Exercises.

(a) Find the fixed points of all the maps stated above, and analyze their stability as function of system parameters. Fixed points are those that do not move, that is $q_{n+1}^0 = q_n^0$, $p_{n+1}^0 = p_n^0$. The stability is checked by find the Jacobian at these points and evaluating their eigenvalues. If the eigenvalues have a real part greater than unity in magnitude, the point is unstable. For 2D area-preserving maps such as above, the eigenvalues that are complex have modulus unity and are elliptic fixed points, while those whose eigenvalues are real (one larger than unity in modulus) are hyperbolic fixed points.

(b) Plot all of the maps above for various parameters and initial conditions. Get a feel for how easy it is to get pretty pictures. You will later see how difficult it is to make sense of them!

(c) Show that translation along p by one unit $S = (p \to p+1,\; q \to q)$ is a symmetry of the standard map. This operation commutes with the map itself, if we call the map M then $M \circ S = S \circ M$. Show that the reflection about the center of the unit square: $(q \to 1-q, p \to 1-p)$ is also a discrete symmetry of the standard map. What about the other maps? What effect do these symmetries have on the way the phase space looks?

In Fig. (2.1) is shown many orbits for the standard map for a few parameter values. The standard map is plotted as if the phase-space was a torus, that is the modulo condition is applied on the momentum as well. This gives us a true picture of what is happening on the cylinder as this unit cell repeats indefinitely along the cylinder. This follows from the fact that translation along p by one unit is a *symmetry* of the standard map (see exercise).

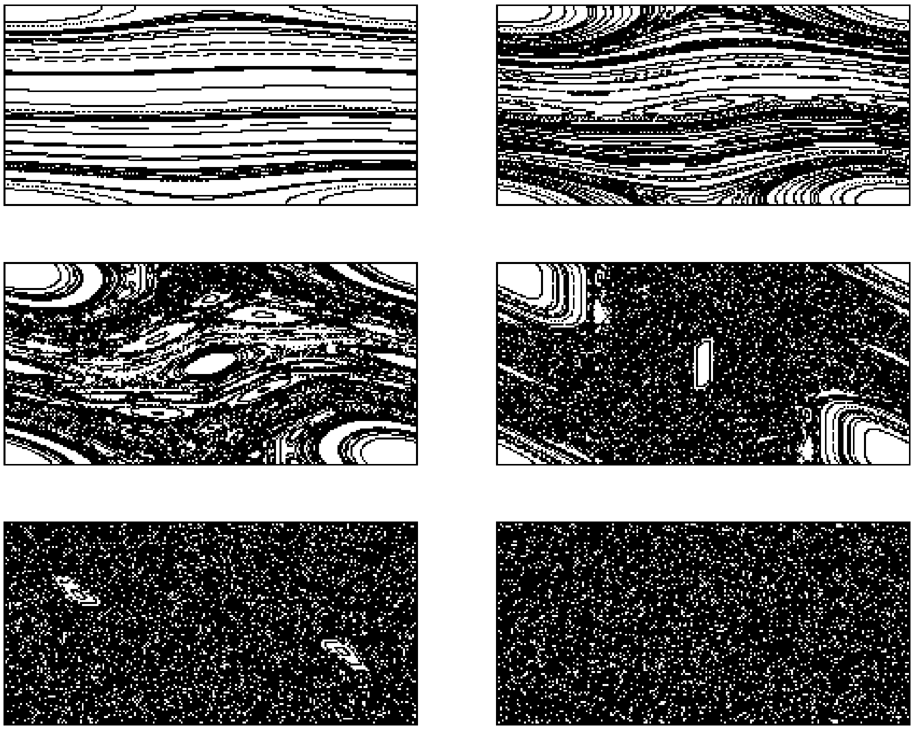

Figure 2.1: The standard map $(q-p)$ plot for parameter values $K = 0.1, 0.5, 1.0, 2.0, 5.0, 7.0$, increasing top left to bottom right for a random collection of 100 initial conditions followed 500 times. In all figures both the q and p directions go from 0 to 1.

The standard map is one of the earliest models to have their *quantum mechanics* studied, a topic we will later touch on. Here it is enough to say that if you plotted the standard map on the cylinder, you will see a diffusion like behaviour in the momentum direction: exactly like a random walker in real space. This diffusion leads to increasing energy to the pendulum indefinitely. It was found that quantizing the system lead to a suppression of this diffusion, which lead to the phrase "quantum suppression of classical chaos".

Problem: **Classical Transport in the Standard Map**:

Iterate the standard map on the cylinder (no mod 1 on the momentum), and plot p as a function of time for various initial conditions for $K >> 6$, say $K = 10$. Notice the similarities between this graph and a 1-d random walker. Then find $< p^2(t) >$, where you first square the momentum and the average ove many initial conditions at a given time. The initial conditions could be for instance chosen to be uniformly spread along $p = 0$, $0 \leq q < 1$. This is the " ensemble" averaging. Similarly find $< p(t) >$ and hence plot $< p^2(t) > - < p(t) >^2$ as a function of time t. You should observe a straight line with a slope approximately proportional to K^2, which can be predicted with simple arguments. However the exact value of the slope (the "diffusion" coefficient) is a very important quantity whose evaluation is a challenging and open problem in general. In particular there are K values where diffusion is violated, due to the presence of small islands, like those seen in the figure for $K = 5$, which don't fully disappear and lead to so-called "anomalous" diffusion, with diffusion mixed with purposeful marching, along so-called "accelerator modes". For further details see the references.

The Fig. (2.1) shows how Hamiltonian chaos develops in typical low-dimensional systems. At $K = 0$, the dynamics is that of a *free rotator*, something called a twist map.

2.2 Twist Map

The map we must begin with is something we can understand:

$$\begin{aligned} q_{n+1} &= q_n + p_n \,(\mathrm{mod}\ 1) \\ p_{n+1} &= p_n . \end{aligned} \tag{2.12}$$

is an example of a *twist map*, in the sense that a radial line from the origin in phase-space will be twisted due to different rotation angles at different momenta (consider momenta to be radial coordinate and q to be the angle). This is a simple map and can be solved exactly:

$$\begin{aligned} q_n &= q_0 + n\, p_0 \,(\mathrm{mod}\ 1) \\ p_n &= p_0 . \end{aligned} \tag{2.13}$$

Exercises.

(a) Derive that Eq. (2.13) is in fact the solution.

(b) Show that if $p_0 = r/s$, where r and s are integers which don't have any common factor (i.e. they are "coprime"), the orbit consists of s distinct points that are periodic. Establish that the role of r is to skip $r-1$ points each time the map is iterated. Thus show that all rational values of p_0 lead to periodic orbits.

(c) Show that if p_0 is an irrational number the orbit will not be periodic. It will in fact uniformly cover the interval $0 \leq q < 1$. It is said to be ergodic on the coordinate axis.

(d) Using the previous parts draw a phase-space portrait of this map. Compare it to the standard map at $K = 0.1$.

We have to see what happens to the twist map when we add terms that make it like the standard. The two types of orbits, the periodic ones and the non-periodic ones fare very differently. But first an objection may be raised that this is not general enough to be of interest: why the twist map? Actually, the twist map is also the *generic form of the PSOS for $d = 2$ separable Hamiltonian systems.*

2.2.1 Twist maps and Poincaré Surface of Sections

This is where the much maligned and largely ignored topic of classical mechanics courses, the Action-Angle variables show up for their wages. An important class of Hamiltonians are close to those that we can solve, so that some sort of perturbation theory can be used, and most of the solvable problems are separable in some appropriate coordinate systems. So let us start with a Hamiltonian of the form:

$$H(q_1, q_2, p_1, p_2) = H_1 + H_2 + V_{12}(q_1, q_2) \tag{2.14}$$

where $H_i = p_i^2/2 + V_i(q_i)$ is the Hamiltonian for the individual degrees of freedom while the potential V_{12} provides an interaction between them. As has been mentioned earlier the generic state of such Hamiltonian systems are chaotic.

However consider first the case $V_{12} = 0$, then what is the PSOS? For then instead of studying the effect of the potential in the full phase-space we can find the effect on the PSOS. Let us consider the case when V_1 and V_2 are confining potentials leading to periodic motions of period $T_i(E_i)$. The period can in general depend on the energies. Let us define the PSOS where $q_2 = 0$ and $p_2 > 0$. Therefore if $(q_1(n), p_1(n))$ is a point on the PSOS we have to simply wait for a time T_2 to get $(q_1(n+1), p_1(n+1))$. Thus schematically we can write the PSOS as:

$$(q_1(n+1), p_1(n+1)) \;=\; f^{T_2}(q_1(n), p_1(n)) \tag{2.15}$$

where f^t evolves phase-space points in DoF 1. This mapping is in general nonlinear and can be quite complicated, although you know that the answer you get will have to be an integrable map, not chaotic! (We know the one constant of motion required, namely E_1.) You will however be able to write the PSOS for some simple cases, such as when the two oscillators are Harmonic, or when one or more of the confining potentials is hard wall like. If both are hard wall like, it is a rectangular "billiard". Thus this form of the PSOS is not revealing and not much perturbation theory can be done with this approach.

However the fact that the mapping is simple in some sense, tells us that there could exist coordinate systems in which the PSOS is simple. The "coordinate system" is the Action-Angle variables that we can get by means of a canonical transformation of the old (q_1, p_1) variables. Recall that if:

$$I_1 = \frac{1}{2\pi} \oint p_1 \, dq_1 \tag{2.16}$$

where the integral is taken over a closed phase space trajectory, then H is only a function of the action I_1, and not its conjugate variable θ_1. I_1 is the area under the curve $H_1 = E_1$ (except for the trivial factor 2π) and, serves like the radial marker in polar coordinates, while the angle tells you where on this constant area curve you actually are. Since the energy surfaces are nested, the actions uniquely

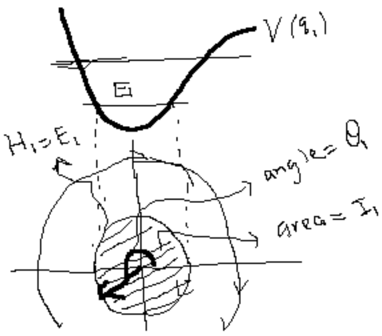

Figure 2.2: Illustrating Action-Angle

determine (in 1D) the orbit, and the angle specifies where on the orbit the phase space point is. See the doodle in Fig. (2.2).

The Action-Angle transformation is a canonical one, in the sense that the the Hamiltonian written in this still generates the equations of morion through the canonical equations of motion, stated right at the beginning. Thus $H_1(I_1)$ is the Hamiltonian and we have

$$\frac{d\theta_1}{dt} = \frac{\partial H_1}{\partial I_1} = \omega_1(I_1) \tag{2.17}$$

$$\frac{dI_1}{dt} = \frac{\partial H_1}{\partial I_1} = 0, \tag{2.18}$$

where $\omega_1(I_1)$ is some in general nonlinear function. Thus in the Action-Angle coordinates, the Action is a constant, while the angle increases linearly, till of course it folds back. Thus in the Action-Angle space the PSOS is:

$$\begin{aligned} \theta_{n+1} &= \theta_n + T_2\,\omega_1(I_n)\,(\mathrm{mod}\ 2\pi) \\ I_{n+1} &= I_n, \end{aligned} \tag{2.19}$$

where we dropped the uniform subscript indicating DoF 1. Thus in these variables the PSOS is a generalized twist map, we had stated earlier the case when $\omega_1(I) = I$, a linear function. In general $\omega_1(I)$ is an increasing function of I, and the above map has all the properties we have already discussed. Notice that $T_2 = 2\pi/\omega_2$, where ω_2 is the corresponding frequency function for DoF 2. Thus depending on whether the *ratio of the frequencies of the two oscillators* is rational or not, we get closed or non-closed orbits. A little thought will show that this is not only reasonable, but in fact evident. The rational ratios result in a continuous family of orbits with the same behaviour, and these lie on *rational tori*, while the other orbits lie on *irrational tori.* On the maps, the rational tori show up as finite periodic orbits, while the irrational ones as continuous curves. The ratio of frequencies is the all important *winding number* and the action of the perturbation on it depends on this number crucially. Thus to summarize, twist maps are the generic form of not only the stroboscopic map of unkicked 1.5 degree of freedom systems (hence simply 1-d systems!), but also the PSOS of separable $d = 2$ DoF systems.

2.3 KAM, Birkhoff, Poincaré

The names in this section title are in the reverse chronological order, the order in which we discuss them, at least state the results to some extent. There are many books that deal adequately enough with these topics, which are really quite complex and form the crux of modern classical mechanics. The oustanding problem of the gravitational three-body problem spurred the major developments of classical mechanics. The question of the stability of the three-body gravitational systems was raised, and nothing less than the eventual stability of the solar system was in question. Today, it is understood that there are many solar system inhabitants that are chaotic, some more so than others. The 16^{th} satellite of Saturn, Hyperion, is known to be a tumbling piece of rock, about 140 Km wide, quite chaotically influenced by the big planet and its large satellites Titan. Its Lyapunov exponent is about $1/100$ (days)$^{-1}$, while its rotation period is 20 days. Two other moons of saturn, Pandora and Prometheus, that are said to "herd" the rings of saturn, are also believed from recent observational data to be chaotic. Pluto has been found to be mildly chaotic and would leave the solar system eventually. Despite all this chaos, it remains a mystery why there is not much more of it in the solar system, after all there are more than three bodies, and three is sufficient for chaos. The apparent stability of the solar system still remains unresolved in its entirety.

2.3.1 Kolmogorov (1954), Arnold (1960), Moser (1962)

KAM theorem address the problem of what happens to an integrable Hamiltonian system that is very slightly perturbed. Just how slightly, just by what perturbation etc. are also stated. KAM asserts that under the conditions of:

1. sufficiently small perturbation,
2. sufficiently smooth perturbation, and
3. for, loosely speaking, sufficiently "nonlinear" integrable Hamiltonians,

"sufficiently irrational" tori survive, that is they are topologically equivalent to the integrable (unperturbed) tori. Suffices to say that a complete proof of this theorem is difficult and involved, as may be judged from the fact it took three great mathematicians to do what Poincaré thought he had done about 60 years earlier: exorcised the small-denominator problem. While K&A were working with Hamiltonian flows, J. Moser addressed the problem from a perturbed twist-map angle.

The persistence of almost straight-line orbits in the first window of Fig. (2.1) corresponding to $K = 0.1$ is an evidence of the working of KAM theorem. In fact lets go over the KAM conditions: is $K = 0.1$ small enough? This is hardly clear, the original proof of Arnold reportedly requiring a perturbation that is less than 10^{-48}! The fact that tori survive almost up to $K = 1$ is certainly not explained by the KAM theorem. These highly non-perturbative regimes are little understood. Is the perturbation smooth? Yes, the sin function is C^{∞}, infinitely differentiable. The Moser proof required that the perturbation be at least C^{333}, again certainly a very odd number! Subsequent developments allow perturbations that are only $C^{2+\epsilon}$. Counterexamples are known for C^2 perturbations, so we may at least require that the perturbation and its derivative is continuous. Is the unperturbed system "sufficiently nonlinear"?

The unperturbed system is assumed to have good action angle coordinates, that is the Hamiltonian $H(I_1, I_2, \ldots, I_d)$ is only a function of actions. The frequencies corresponding to the actions are $\omega_i(I_i) = \partial H/\partial I_i$. The condition of nonlinearity actually requires that these frequencies not be degenerate. This can happen if the frequencies don't depend on the actions at all. This happens in the case of the harmonic oscillator. Recall that the frequency does not depend on the energy, and hence on the action, indeed the Hamiltonian is $H = \omega I$ for the 1-D oscillator. More generally it is required that

$$\mathrm{Det}\left|\frac{\partial \omega_i}{\partial I_j}\right| \neq 0.$$

Translated into English, this means that given a set of frequencies they locally determine a set of actions and hence we can specify the tori based on the frequency functions. The implicit function theorem (see books on calculus) guarantees this provided the determinant condition holds. Just think $d = 1$, this is not valid for the harmonic oscillator but valid for generic potentials. This explains why we were only "loosely speaking" when we said that the unperturbed system is "nonlinear", it has to be nonlinear and nondegenerate.

The unperturbed system for the standard map is the twist map, and the angle is q, while the action is p and we see that the frequency function is $\omega(I) = I$, therefore the unperturbed system is not degenerate, and the KAM conditions are satisfied. But note that the condition of sufficient nonlinearity excludes the applicability of KAM to many physically interesting problems, including perturbed Harmonic oscillators. Non-KAM behaviour may or may not be seen if any of these conditions are violated. Non-KAM systems are even less understood, although examples abound.

2.3.2 Poincaré and Birkhoff.

The theorem associated with the French founder of modern dynamics, Henri Poincaré and his American follower Georg Birkhoff, explains the fate of *rational tori.* While KAM said "nothing much" happened to the irrational ones, the rational ones undergo dramatic change, it is an example of a bifurcation, where marginal orbits are replaced by a chain of stable (elliptic) and unstable (hyperbolic) orbits.

The proof the PB theorem is in fact much easier than the KAM and is largely geometrical. Again, we simply state the result. Let the unperturbed rational tori have a winding number $\omega = r/s$. Then on perturbation this will be replaced by a chain of alternating hyperbolic and elliptic periodic orbits of period s, with a winding number of r/s. It should be emphasized that the unperturbed rational tori supports an *infinity* of periodic orbits of period s (the whole line, in the case of the twist maps), but the perturbed system has only *one.* around the elliptic periodic orbits there are closed stable orbits, while around the hyperbolic orbits something very strange occurs that eventually matures to chaos.

The PB scenario for rational tori, is a frequency matching commensuration condition and this is therefore called a "resonance", an $r : s$ resonance is said to be occur. The smaller the s is, that is smaller the period, on perturbation the larger is the elliptic or the resonance zone. As perturbation increases the resonance region, which resemble the simple pendulum phase space, grows. The fattening resonances overlap and this gives rise to global chaos in one of the scenarios of Hamiltonian chaos, (due to Chirikov, the "chirikov overlap criterion").

Let us take the example of the standard map, Eq. (2.9). At $K = 0$, the twist map has a $0 : 1$ resonance all along the line $p = 0$. This entire line is a line of *fixed* points, that is points of period 1. For $K \gtrsim 0$ there are only *two* fixed points, the stable one at $(0, 0)$ and the unstable one at $(0.5, 0)$, and the region becomes like a simple pendulum. For increasing K this region grows, till it eventually collapses, as can also be seen from the Fig.(2.1). The PB scenario can be seen in abundance in the standard map.

But what gives rise to the splatter of points? If KAM and PB were the whole story then the phase space should not look anything much more complicated than that of a pendulum. One key factor in the differences between KAM and PB was the rationality of otherwise of the winding number; but on the real line, *both the sets of rational and irrational numbers are dense.* This means that arbitrarily close to any point we can find a rational number, we can also find an irrational number. This means that the resonances are also dense. However each resonance has some width, and therefore we can expect that resonances will generically intersect even for very small perturbations, and result in complex dynamics.

Therefore it seems reasonable that *chaos begins on the boundaries of resonances*, especially in the neighbourhood of the hyperbolic periodic orbits.

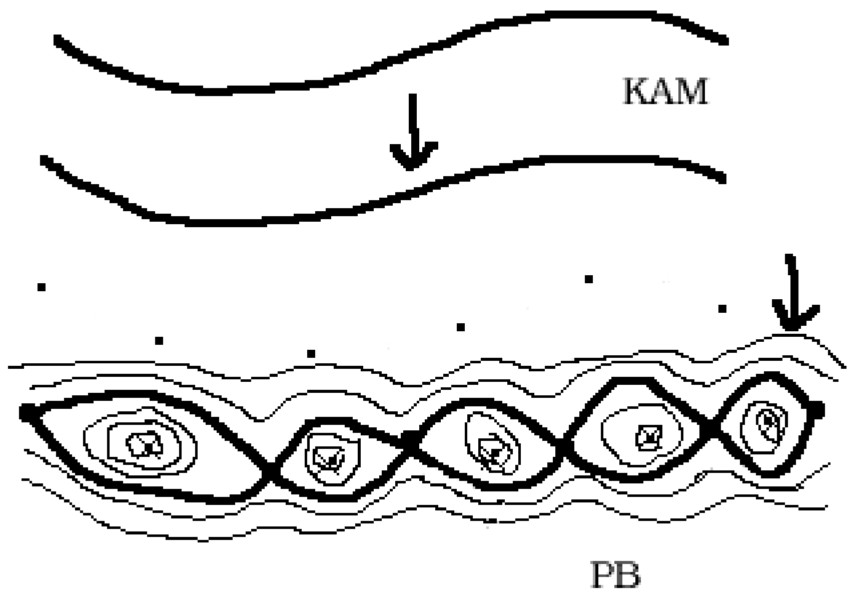

Figure 2.3: A doodle of the KAM and PB theorems. An irrational tori survives the perturbation, while a rational torus (here $s = 6$) breaks into a resonance chain. Shown here as a sequence of six dots is only a representative orbit of the unperturbed rational tori, which has an infinite number of them.

2.3.3 The Homoclinic Tangle or $1 \Rightarrow \infty$.

In a 2D area-preserving map the neighbourhood of a hyperbolic periodic point, say x^* contains the seeds of chaos. There are a set of points that approach the hyperbolic points ever more closely, while there is a set that was very close to it in the ever receding past. The first set is called the *stable manifold*, denoted $W^s(x^*)$, while the second is called the *unstable manifold*, denoted $W^u(x^*)$. In general These "manifolds" (just think of manifolds as smooth nonintersecting curves for our purpose) are a collection of an infinity of orbits.

Exercise.

FIND THE EQUATION FOR THE STABLE AND UNSTABLE MANIFOLDS OF THE SIMPLE (UNFORCED!) PENDULUM.

Firstly, the stable and unstable manifolds do not intersect each other. Remember that the manifolds can consist of many orbits. Only in a $d = 1$ Hamiltonian flow the manifolds coincide with orbits themselves. Therefore for such systems (or in general flows in 2D phase space) the fact that the stable and unstable manifolds do not self-intersect is a direct consequence of uniqueness of solutions, something we have already seen. For higher dimensions, the manifolds are themselves not orbits, this argument is not directly applicable.

However let us say that an unstable manifold intersects itself as in Fig. (2.4), then from *continuity* of the transformation ("nearby points go to nearby points") it follows that such a self intersecting point

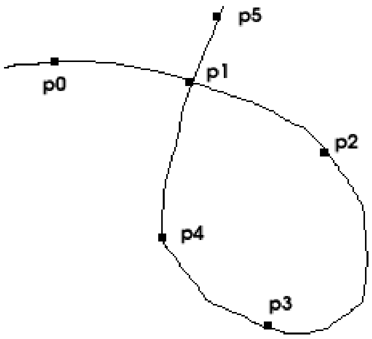

Figure 2.4: If an unstable (or stable) manifold intersects itself ...

must be mapped to another self intersecting point, which gets mapped to another one and so on ad-infinitum. But also the self-intersecting point must have *come* from such a point etc. to the distant past when the orbit was arbitrarily close to x^*. Thus self-interesecting points must accumulate to this point. This is not possible, as from a fundamental theorem of dynamical systems, the Hartman-Grobman theorem, the unstable manifold is tangential to an eigenvector of the linear stability matrix at that point (the *local* unstable manifold). Similar argument for the stable manifold, reversing time now, and going forward. Thus indeed the sets of stable and unstable manifolds are *manifolds*.

This nonintersection also follows from invertibility and continuity, as two points cannot be mapped to the same. Refer to the Fig. (2.4), where we have marked a segment of an orbit as $p0 \to p1 \to p2 \to p3 \to p4 \to p5$. The point $p4$ "jumps" over the point $p1$ to reach $p5$ (remember this is a map). However from continuity the segments of orbits in $p0 - p1$ gets mapped to $p1 - p2$ etc., and therefore there must be some point in the segment $p3 - p4$ that gets mapped into $p1$. But $p1$ came from $p0$ and cannot have two immediate predecessors and therefore the envisioned scenario is prohibited.

If the unstable manifolds belong to two different periodic points, then they clearly cannot intersect each other, as in the distant past there would be a conflict of interest and similarly for stable manifolds in the distant future.

Surprisingly, the stable and unstable manifolds *can* and do intersect each other, with dramatic consequences. If they belong to the same orbit such an intersection is called *homoclinic* and if they belong to two different orbits it is *heteroclinic*. Neither of the two arguments against intersection hold here. The first argument about one intersection implying another and so on also holds here. If a point belongs to both the stable and unstable manifold, its distant past and distant future is sealed, it must be in the vicinity of x^*, therefore all its iterates also have this property and hence lie on intersections. Combined with the area preserving nature of the transformations (Fig. (2.6), the homoclinic intersections give rise to a complex network. Phase space points that are in this network behave in a apparently random manner, although their dynamics is completely deterministic.

If you think that the homoclinic tangle is a mess, you are in good company,

"*It may be futile to even try to represent the figure formed by these two curves and their infinite intersections, each of which corresponds to a doubly asymptotic solution [to the equations of motion]. The intersections form a kind of trellis, a tissue, an infinitely tight lattice; each of the curves must never self-intersect, but it must fold itself in a very complex way, so as to return and cut the lattice an infinite*

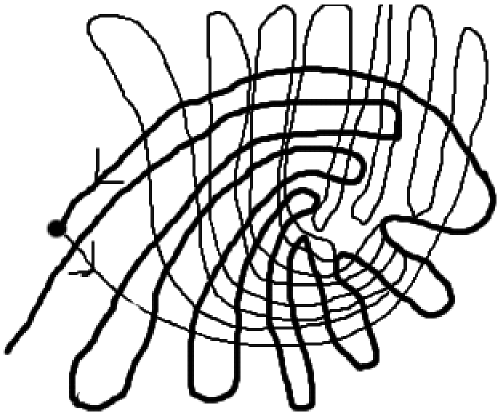

Figure 2.5: A homoclinic tangle doodle. The dark line is the stable manifold, while the other is the unstable one. Notice how they have to arrange themselves in order not to self intersect.

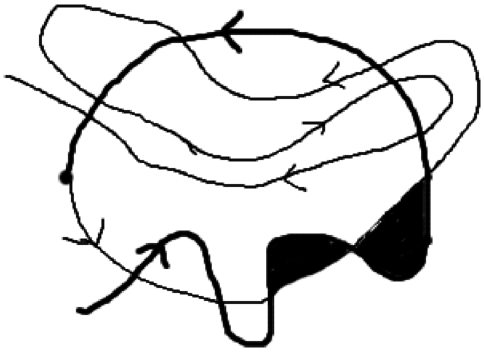

Figure 2.6: The shaded areas are successive mappings.

number of times. The complexity of this figure is so astonishing that I cannot even attempt to draw it. Nothing is more appropriate to give us an idea of the complexity of the three-body problem and in general of all the problems in Dynamics where there is no uniform integral."

Henri Poincaré, 1899 on homoclinic orbits.

Exercise. *Draw the homoclinic tangle. Draw an heteroclinic tangle. Does area conservation along with nonintersection lead to a contradiction? Won't the shaded areas such as in Fig. (2.6) not be able to intersect itself, yet not go away and therefore fill a finite area infinitely, which is clearly absurd. The answer is in this figure itself. Go figure.*

The neighbourhood of each of the hyperbolic points in the PB scenario is in fact a homoclinic tangle. To understand the tangle is to understand the genesis of Hamiltonian chaos. As physicists we look to abstract this object, make it so simple that it cannot get simpler without destroying what we study. The dissipative abstraction of the tangle is called the Smale (pron.: Small a) horseshoe, after its originator Stephen Smale and the shape of the areas on iterations, horseshoes. We will however look at a simple model that is still globally area preserving: the baker map, named after the generic humble baker who quite oblivious of the homoclinic tangle is engaged in baking the daily bread.