Lecture 3

3.1 The Baker's Map

"The baker's map is the harmonic oscillator of chaos"
–Predrag Cvitanovic.

On the lightly floured surface, flatten the dough slightly into a disk-shape. Use the heels of your hands to PUSH the dough away. Pick up the edge furthest away from you and FOLD it toward you, sliding the dough back to its original spot on the counter. TURN the dough a quarter-turn. Vigorously repeat "push, fold, and turn" steps.

– From www.baking911.com. (without permission).

So whats all this fuss about? The baker does it every day, and while making chappathis we adopt similar strategy to mix thoroughly the *atta* with the water. Looking at the instructions for baking we see two crucial ingredients, PUSH, and FOLD. Pushing provides the instability, while folding provides the confinement, the classic "stretch and fold" mantra of chaos. (For the moment we ignore the instruction of the baker to "turn"). We also assume that the dough is 2D and a perfect square!

BAKER MAP.

$$\left.\begin{aligned} q_{n+1} &= 2\,q_n \\ p_{n+1} &= p_n/2 \end{aligned}\right\} 0 \le q_n < 1/2; \quad \left.\begin{aligned} q_{n+1} &= 2\,q_n - 1 \\ p_{n+1} &= (p_n+1)/2 \end{aligned}\right\} 1/2 \le q_n < 1. \tag{3.1}$$

The baker map is area preserving, its Jacobian is simply $\mathrm{diag}(2, 1/2) = \{(2, 0), (0, 1/2)\}$ everywhere, the eigenvalues are 2 and 1/2. The map is said to uniformly hyperbolic, the stretching is the same throughout phase space. Hamiltonian systems of d DoF are always characterized by a maximum of d pairs of stretching and compressing directions as the overall area (and in fact various other quantities known as Poincaré Cartan integral invariants) has to be preserved. This leads to the sum of the d Lyapunov exponents being zero. Thus the lyapunov exponents for the baker map are $\log(2)$ and $-\log(2)$. Thus with a positive nonzero Lyapunov exponent the map is a candidate for chaos. It is in fact a paradigm of chaos, and this becomes clear if we work in the binary representation of numbers.

Let (q_0, p_0) be an initial phase space point that is evolved by the baker map in Eq. (3.1). How is the phase space point after n iterations (q_n, p_n) related to the initial conditions? In short can we *solve* the baker map? If you look more closely at the baker map, you realize that the evolution of q has nothing to do with p, while the evolution of p is dependent on q. Thus we could solve for q_n for instance, and in fact the q map is known as the doubling or the sawtooth map. It is simply $q' = 2q \,(\mathrm{mod}\, 1)$, whose solution is $q_n = 2^n\, q_0 \,(\mathrm{mod}\, 1)$. What have we achieved? Actually not much. Although we have "solved" for q_n the solution does not leave us much wiser. This is a good illustration that *chaos is not about the lack of analytical solutions.* Chaos has little to do with our labels of "numerical", "analytical" etc., it is a statement about dynamics.

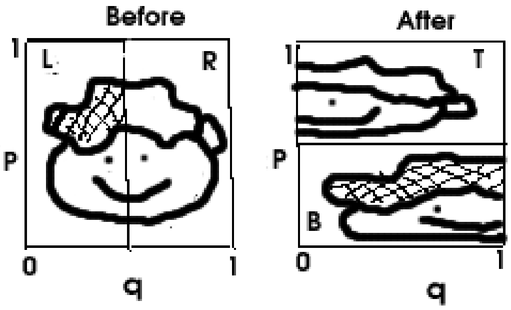

Figure 3.1: The dough before and after. Notice how the baker's smile becomes a bit wane, as his face is stretched in the q direction and compressed in the p, both by a factor of 2, cut in the center and stacked with the right half on top of the left, so that $L \to B$ and $R \to T$. It is a wonder that the baker is still smiling at all.

To "understand" the baker any further, we need to do something more than solve it. Let the phase space points be specified in the binary representation, say as $q_0 = 0.a_0a_1a_2\ldots$ and $p_0 = 0.a_{-1}a_{-2}a_3\ldots$, where $a_i = 0, 1$. Recall that this means

$$q_0 = \frac{a_0}{2} + \frac{a_1}{2^2} + \frac{a_2}{2^3} + \ldots; \; p_0 = \frac{a_{-1}}{2} + \frac{a_{-2}}{2^2} + \frac{a_{-3}}{2^3} + \ldots. \tag{3.2}$$

Note that $a_0 = 0$ if q_0 lies in left half L and it is 1 otherwise. Thus $q_1 = 2\,q_0 - a_0$ whether $a_0 = 0\,\text{or}\,1$. Similarly $p_1 = (p_0 + a_0)/2$ in either case. We get

$$q_1 = \frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \ldots; \; p_1 = \frac{a_0}{2} + \frac{a_{-1}}{2^2} + \frac{a_{-2}}{2^3} + \ldots. \tag{3.3}$$

If we write Eq. (3.2) as a *bi-infinite* sequence of the following form

$$(p_0|q_0) = \ldots a_{-3}a_{-2}a_{-1} \circ a_0a_1a_2 \ldots \tag{3.4}$$

then

$$(p_1|q_1) = \ldots a_{-2}a_{-1}a_0 \circ a_1a_2a_3 \ldots \tag{3.5}$$

The notation $(p|q)$ is to indicate a phase space point, the right side is to be thought of symbolically, with the small circle separating the p in the left from the q on the right. Thus given the right side, we can read off q and p. Thus the action of the baker dynamics is transparent. It shifts the entire sequence of a_i to the left by one place. Iterating the map simply shifts it further and futher. Thus the small circle, dividing q from p, also acquires the significance of the *present moment.* This dynamics is known as a *left shift.* Although this appears to be very special it is some form of left shit on possibly infinite symbols (not just 0 and 1) that underlies chaos, including that found in many body problems. The left shift is in many senses the heart of chaos, Hamiltonian or otherwise.

This representation then reveals many features of the baker map:

1. Proliferation of errors. The left shift reveals the alarming nature of chaos and eventually the limitations of mathematical modelling for prediction. We would know the initial conditions only up to some precision, say we have an accuracy of $10^{-20} \approx 2^{-67}$, that is we know $(a_0, a_{\pm 1}, \ldots, a_{\pm 66})$ and a_{-67}.

The rest of the binary expansion coefficients are inaccessible to us. On iterating the baker map, we lose one bit of information every time, and after 66 times the following is the case:

$$\begin{aligned}(p_0|q_0) &= [u]\, a_{-67} \ldots a_{-1} \circ a_0 \ldots a_{66}\, [u] \longrightarrow \\ (p_{66}|q_{66}) &= [u]\, a_{-67} \ldots a_0 \ldots a_{65} \circ a_{66}\, [u].\end{aligned} \tag{3.6}$$

We have indicated the any unknown bit strings uniformly as $[u]$. The initial area of ignorance is a square of size $2^{-67} \times 2^{-67}$, whose left-bottom corner is at $(q = 0.a_0 \ldots a_{66},\ p = 0.a_{-1}a_{-2} \ldots a_{-67})$.

What do know about $(p_{66}|q_{66})$? About q_{66} we only know the first significant bit which is a_{66}, we are at the edge of our knowledge. Thus all we can say about q_{66} is whether it is in the left half ($a_{66} = 0$), or the right half ($a_{66} = 1$) of the square. We seem to have compensated for our q losses in p, we now know it to a *greater(!)* accuracy of about 10^{-40} now. After one more iteration we have no clue where q_{67} is, we can only make the trivial, but consoling, statement that it is somewhere in the square, we are not losing any dough, and the accuracy of p is at its best. The area of ignorance is now the rectangle of size (1×2^{-134}), with the left-bottom corner being at $(q = 0, p = 0.a_{66}a_{65} \ldots a_0a_{-1} \ldots a_{-67}.)$

What next? Only *now* does chaos rear its head. The situation described in the previous paragraph is really only a consequence of instability, the PUSHING action of the baker. It is true for unstable nonchaotic systems, like a ball rolling down a hill, as well. the unknown bits of q now become the *most significant* bits of p due to FOLDING. Now we have a peculiar situation that we know some of the less significant bits of p, but the most significant ones are being lost, or rather are being taken over by the unknown bits of q. What is the error in p? Remember that the error in q is now the maximum possible, namely 1.

Lets take stock after $n = 68$ steps.

$$(p_{68}|q_{68}) = [u]a_{-67} \ldots a_{66}u_1 \circ [u], \tag{3.7}$$

where we denote by u_k, one of the 2^k unknown bit strings of length k, u_1 could be 0 or 1. Thus the area of ignorance is now *two* rectangles, each of area 1×2^{-68}, one whose left-bottom corner is at $(q = 0,\ p = 0.0a_{66} \ldots a_{-67})$ and the other at $(q = 0,\ p = 0.1a_{66} \ldots a_{-67})$. After one more time step there are *four* areas of ignorance of size 1×2^{-69} etc. Thus it is clear that the area of ignorance remains a constant, after this *is* an are preserving transformation! However the area of ignorance after a critical time, which depends on the original area, starts taking on such complex and proliferating branches, typically exponentially large in number, that it loses the typical meaning of "error bars" anymore.

Naturally a picture is in order here. See Fig. (3.2) for an illustration where the initial area of ignorance is pretty large, being one quarter of the square (some really bad equipment here!), this size meaning we don't wait inordinately long for things to happen. The evolution of this area is $[u]0 \circ 0[u] \to [u]00 \circ [u] \to [u]00u_1 \circ [u] \to [u]00u_2 \circ [u]$. The two possibilities for u_1 is reflected as the two horizontal strips at $n = 2$ (the one at the top is $u_1 = 1$, while the one below is $U_1 = 0$); similarly the four possibilities for u_2 results in four strips at $n = 3$.

Thus while the area of ignorance remains constant, its shape changes in such a way that its boundaries are not practical to keep track of any longer. In the case of the baker, the error along the q direction initially increases till it reaches a constant (1), while the error along p decreases till it reaches a constant, such that the total area is always preserved. However these errors are not practical error bars any longer as the region is not a simple one. In fact if we have as a possibility on measurement a scatter of points, we are sure about the result being one of them, then the area of error is zero, but this is quite absurd as a practical source of error measurement.

On the other hand, the baker map is very special in that it singles out q and p axes, in general the error will increase along a direction that itself changes in a complex way along the orbit, this infact

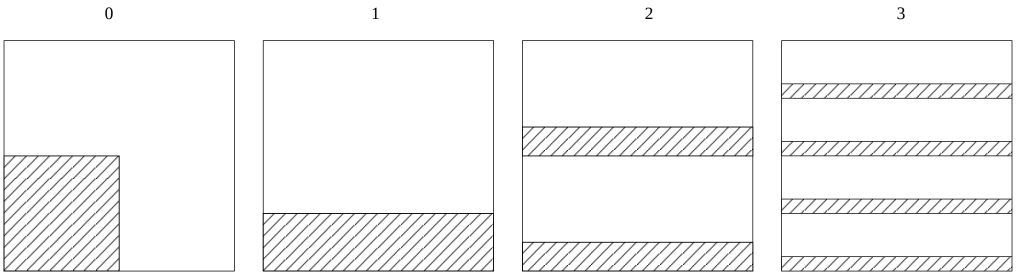

Figure 3.2: Evolution of an area in the baker map

being the unstable manifold, and the error will decrease along the stable manifold. We have seen that the geometry of these is so complicated that all practical observables will experience an increase in the error and it is quite right to say that *errors proliferate, typically exponentially, in the case of chaotic systems.*

Sensitivity to Initial Conditions. Closely related to the proliferation of errors is this property that is often taken to be synonymous with chaos. Two initially close points will share a very long common number of bits in both p and q, but the differences will come to the fore, exponentially fast, they will fly farther apart along q, while coming together along p till they separate also on p.

Mixing. The evolution of areas shown above also illustrates the important property of mixing, the shaded region has the same area but tends to get thoroughly mixed in the background. This is of course why the baker was interested in this transformation in the first place. Mixing was properly described mathematically by Hopf, who also invented the baker map. It is characterized by certain numbers that describe how fast it occurs, the so called Ruelle-Pollicot resonances, something we will not expand upon here.

Periodic Orbits Everywhere. From pictures such as we saw for the kicked pendulum and the standard map, we may think that chaos is some formless mess. This is not necessarily the case, in this mess there is a backbone of periodicity, a periodicity that is more prolific than even for regular systems. Consider the fate of points such as

$$(p_0|q_0) \; = \; \ldots u_k \, u_k \, \circ u_k \, u_k \ldots \tag{3.8}$$

where u_k is any binary string of length k. It is then clear from the left shift dynamics that $(p_k|q_k) = (p_0|q_0)$, that is this initial point comes back to itself after k steps. It also immediately follows that after any multiple of k the point recurs, in short we have a *periodic point* of period k. As there are 2^k possible u_k strings for any k, there are exactly 2^k possible periodic points for the baker map of period k. The repeating string is often denoted by an overbar, thus $(p_0|q_0) = \overline{u_k} \circ \overline{u_k}$.
For example the point $\overline{01} \circ \overline{01}$ is a periodic point of period 2, and in usual decimal representation it is $(q_0 = 1/3 \, , p_0 = 2/3)$.
Exercise.

1. *Enumerate all periodic points of period* 1, 2, 3 *and* 4, *writing them out in binary and decimal forms. Note how period* 4 *points include period* 2 *points and how all of them include period* 1 *(fixed) points.*

2. *Prove that the decimal representation of* $(p_0|q_0) = \overline{u_k} \circ \overline{u_k}$, *is*

$$q_0 = d(u_k)/(2^k - 1),\ p_0 = Rd(u_k)/(2^k - 1). \tag{3.9}$$

If the string $u_k = a_0 a_1 \ldots a_{k-1}$ *then* $d(u_k) = 2^{k-1}a_0 + 2^{k-2}a_1 + 2^{k-2}a_2 + \ldots + a_{k-1}$ *is the value of the binary string* u_k, *and* $Rd(u_k) = 2^{k-1}a_{k-1} + 2^{k-2}a_{k-2} + \ldots a_0$ *is the value of the string that is obtained by reversing the string* u_k, *with the least significant bit becoming the most significant bit etc. Note that this is different from the bit-reversal operation that interchanges the 0 and 1 bits. Getting Rd from d is quite a nontrivial, nonlocal, operation that has in it an element of the complexity of the baker map. In an (apparently) totally unrelated event, this operation is central to the Fast Fourier Transform (FFT) and occurs as the scrambling of the output frequencies.*

The exercise shows how to find the periodic orbits. Where are they? The *periodic orbits are dense*, that is *arbitrarily close to any phase space point there is some periodic orbit.* The proof is also easy from the left shift dynamics. If we take as the random point $(p0|q0) = \ldots a_{-1} \circ a_0 a_1 \ldots$, and we want to find a periodic point within a radius $\epsilon < 2^{-K}$ for some large and positive K, we will consider the periodic point with the repeating string $u_{2K} = a_0 a_1 \ldots a_K a_{-K} a_{-K+1} \ldots a_{-1}$. At $n = 0$ this will be within 2^{-K} distance from the chosen random point, but it will be an periodic orbit of period $2K$. Thus it is clear that periodic orbits are dense.

Exercise.

Show that there is a dense orbit in the baker map. This is an orbit such that it comes arbitrarily close to any other phase space point. In fact dense orbits are dense in phase space. Dense orbits imply and are implied by ergodicity. Roughly speaking ergodicity guarantees that time averages can be calculated by suitable phase space averages, because any typical trajectories explores all the nooks and corners of the phase space, so to speak.

Stable and Unstable Manifolds. Homoclinic and Heteroclinic Points. For any point or orbit in the baker map the stable and unstable manifolds are easily found using the left shift. For instance for the fixed point at the origin any point whose position (in binary) ends in an infinite string of zeros is in the stable manifold. In general for a periodic orbit of period k characterized by the string u_k the following is easily seen:

1. Stable Manifold: $something \circ somethingelse\,\overline{u_k}$.
2. Unstable Manifold: $\overline{u_k}\,something \circ somethingelse$.
3. Homoclinic points: $\overline{u_k}\,something \circ somethingelse\,\overline{u_k}$.
4. Heteroclinic points: $\overline{u'_l}\,something \circ somethingelse\,\overline{u_k}$.

Here the heteroclinic points approach the periodic orbit characterized by u_k in the infinite future, but came from a different periodic orbit characterized by a different string u'_l, which could also have a different period $l \neq k$. *something* and *somethingelse* are arbitrary strings of 0 and 1. For example $\overline{01} \circ 10\overline{01}$ is a point that is homoclinic to the period 2 point $(2/3|1/3) = \overline{01} \circ \overline{01}$, while $\overline{101} \circ 001011\overline{10}$ is an heteroclinic *connection* between the period 2 orbit $\overline{10} \circ \overline{10} = (1/3|2/3)$, to which it comes arbitrarily close in the future and the period 3 orbit $\overline{101} \circ \overline{101} = (5/7|5/7)$, from which it deviated arbitrarily closely the sufficiently distant past.

Therefore if both p and q coordinates were rational numbers, then the orbit is either itself a periodic orbit, or is homoclinic to a periodic orbit, or is an heteroclinic connection between two periodic orbits. All other phase space points densely fill the phase space. We also know that irrational numbers are of measure one in the real number line. This means that if we were to select at random a real number it is, with probability 1, an irrational number. Thus although *periodic orbits are dense in the phase space, they are of measure zero.* While these statements are very easy to see for the baker map, they are also believed to be widely applicable for any Hamiltonian system. Rigorous proof of these, even for particular systems is very difficult.

Prime Periodic Orbits. Counting. If $p(n)$ denotes the total number of periodic points of period n, for the baker map we found that

$$p(n) \;=\; 2^n \;=\; e^{\ln(2)\,n}. \tag{3.10}$$

There are an exponentially large number of periodic points, for example there are about a million periodic points of period 20, all jostling for space on the unit square. The rate of growth of the number of periodic points with period which is $\ln(2)$ for the baker map is an important dynamical indicator, and in general this quantity is called the *topological entropy.*

Each such periodic point lies on a *periodic orbit* of period n. Thus we may say that there are $p(n)/n$ periodic orbits. We see that this is not exactly true as the periodic point may also lie on orbits whose periods are *divisors* of n. For instance all the fixed points are of period n for any n. All period 2 orbits are orbits of even periods etc. So we may ask how many genuine periodic orbits of period n are there? By genuine we mean those that don't have a lesser period that is a factor of n. Such orbits are called *primitive* periodic orbits. Points on the primitive orbits are primitive periodic points. Let $P(n)$ denote the number of primitive periodic points of period n, will this also increase exponentially with the period? This is a more harder and interesting question, whose answer is yes. To evaluate $P(n)$ we encounter an interesting function of number theory that seems as chaotic as the baker map, the Möbius function. A little thought shows that:

$$\sum_{k|n} P(k) \;=\; 2^n, \tag{3.11}$$

where the notation $k|n$ is for those k that divide n including 1 and n. The inversion of such formulae for arithmetic functions ($P(n)$ is an arithmetic function, it maps integers to integers) is achieved by the Möbius inversion formula.

Möbius Inversion Formula. If

$$\sum_{k|n} f(k) \;=\; g(n), \;\text{then} f(n) \;=\; \sum_{k|n} \mu(k)\, g(n/k). \tag{3.12}$$

Here $\mu(k)$ is the Möbius function defined as:

$$\mu(n) = \begin{cases} 0 \;\text{ if } n \text{ has one or more repeated prime factors.} \\ 1 \;\text{ if } n = 1 \\ (-1)^k \;\text{ if } n \text{ is the product of } k \text{ distinct prime factors.} \end{cases} \tag{3.13}$$

A proof the inversion formula is found in many standard books on number theory.

Using the Möbius inversion formula we can write

$$P(n) \;=\; \sum_{k|n} \mu(n/k)\, 2^k. \tag{3.14}$$

While this cannot be summed up to any simpler form, it provides an efficient way of calculating $P(n)$. It is found that $P(n) \sim 2^n$ also, so that the number of periodic orbits $\sim 2^n/n$. This generalizes to the important rule:

$$\# \text{ of Periodic Orbits of Period } T \sim \frac{e^{hT}}{T}. \tag{3.15}$$

The constant h is the topological entropy.

Exercise. *Find $P(1), \ldots, P(10)$. Write a computer program to find $P(n)$ and plot $\ln(P(n))$ vs n, verify that it is a straight line with a slope that approaches $\ln(2)$ for large n. The author found the "Mathematica" software ideal to do such calculations, as this package has in-built in it arithmetical functions such as the Möbius.*

In Fig. (3.3) we see all the periodic orbits of periods 5, 10 and 15, there are 32, 1024 and 32768 points in each respectively. It is also clear that the periodic orbits are *isolated*, they do not form continuous families. This distinguishes chaotic systems from most regular ones. The stability of periodic orbits is of course of central importance. In the baker map all the isolated periodic orbits are *unstable*. If the baker map were not exactly solvable in terms of an explicit left shift, we would not have a ghost of a chance finding "long" periodic orbits even of length as small as 15. Numerical errors of deviation from the periodic orbit will take us far afield.

The arrangement of periodic orbits in the phase space is important, and for area preserving maps and for PSOS of $d = 2$ Hamiltonian systems it is believed that they are uniformly spread over the phase space. The periodic orbits of the baker map hold out intriguing patterns, as yet unexplored, of course to the best of the authors' knowledge.

Exercise.

1. *Prove that all the periodic orbits of the baker map are uniformly unstable with a Lyapunov exponent of* $\ln(2)$.
2. *Devise an algorithm to find all the periodic orbits of a given period in the baker map.*
3. *Devise an algorithm for only primitive periodic orbits.*

Exercise. The 3D baker map.

While the baker map discussed above is very illuminating from the point of view of 2 DoF Hamiltonian system, it unfortunately must leave a real baker bewildered, no piece of dough, however flat is in flatland. For the baker map not to get pie on its face in the pastry industry, we propose that you study the 3D version. Indeed what we will do is to take the "TURN" exhortation of the baker, from the quotation at the beginning of the baker discussion, to heart. Thus devise a model from a cube into itself, that incorporates the "PUSH", "FOLD" and "TURN" instructions and analyze it for chaos. You may take the FOLD step to be as in the 2D baker, to be a "cut and translate" step.

As can be seen from above the baker map has many features of chaotic systems in a completely transparent manner and helps us understand features of what is called "hard chaos". In particular the dense set of unstable periodic orbits plays a very important role in both classical as well as semiclassical

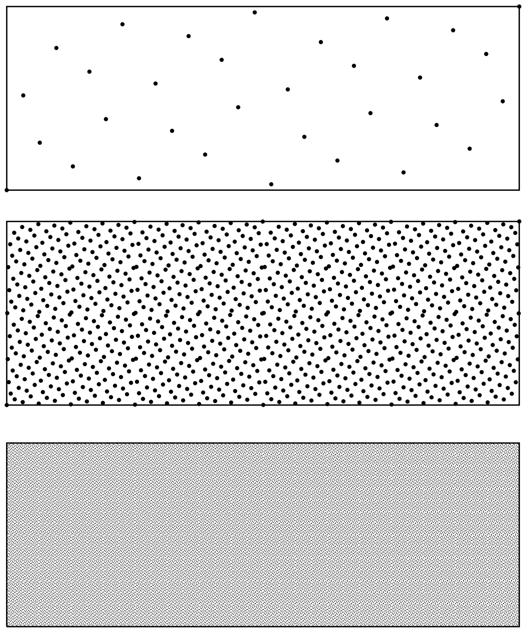

Figure 3.3: All periodic points of the baker map of periods 5, 10 and 15 from top to bottom. The plots are all on unit (q, p) squares.

mechanics. The fact that they exponentially proliferate $P(T) \sim \exp(hT)/T$ and are uniformly spread in phase space are also of crucial import.

While there are more aspects of this map that can be usefully studied, we turn to another system which is better suited for our purposes now, the so called " cat map". We wish to turn attention away from individual orbits to collections of orbits, or densities. The relevant classical mechanics equation will then be the Liouville equation rather than Hamiltonian equations. These two formulations, while mathematically distinct, are of equal importance, and are complementary descriptions of the same dynamical system.

Statistical mechanics deals with collections of orbits, or ensembles, and during the early days of statistical physics, ergodicity was discussed extensively. During the momentous developments of the last century, these questions were relegated to relatively dark corners, as important models of equilibrium statistical physics, such as the Ising model, took the centerstage. There is now renewed interest in the foundations of nonequilibrium statistical physics and its relationship, if any, with dynamical chaos. For instance one issue that comes up with regularity is whether chaos can explain the lack of time-reversibility in the macroscopic world, despite the fundamental equations of motion possessing this property.

3.1.1 The cat map

The cat map was introduced by Arnold and Avez (1967) in their study of ergodicity in classical mechanics. It is also a 2D area preserving map defined on the unit square or torus. Unlike the baker map it is a smooth system, no discontinuity exists. It is linear, the nontriviality of the map arises solely from the re-injection of trajectories thanks to the periodic boundary conditions. It is given by the map:

$$\begin{aligned} q_{n+1} &= a\, q_n \;+\; b\, p_n \,(\text{mod } 1) \\ p_{n+1} &= c\, q_n \;+\; d\, p_n (\text{mod } 1) \end{aligned} \tag{3.16}$$

There are four parameters (a, b, c, d) and the map will be continuous if these are integers, in fact they are smooth dynamical systems in this case. We also assume that $ad - bc = 1$, which makes them area preserving and if $|tr| = |a + d| > 2$ then the map is also hyperbolic and completely chaotic. The map with $(a = 2,\ b = c = d = 1)$ is usually referred to as the cat map, or the Arnold cat map, but we will call this generalization as the family of cat maps. The two eigenvalues of the matrix $\{(a, b), (c, d)\}$, $\lambda_{\pm} = (tr \pm \sqrt{tr^2 - 4})/2$ provide the expansion and contraction rates, and the positive Lyapunov exponent is $\ln(\lambda_+)$.

The cat map is so called, because its inventors illustrated its action on a set of points by displaying what happens to its image as the map is iterated. However since the cat should hardly be credited for the development of chaos in this world, we illustrate it with another picture instead. See Fig. (3.4).

Consider what happens to a collection of initial conditions described by the density function $\rho(q, p; 0)$. This means that $\rho(q, p; 0)\, dq\, dp$ is the fraction of initial conditions in the area $dq\, dp$ around (q, p) at time 0. After a time n the initial condition is at say (q_n, p_n) and the area is now $dq_n\, dp_n \;=\; dq\, dp$, due to area preservation. Thus we have that

$$\rho(q_n, p_n; n) \;=\; \rho(q, p; 0). \tag{3.17}$$

Symbolically if we write $(q_n, p_n) = f^n(q, p)$ as the time evolution we get

$$\rho(q, p; n) = \rho(f^{-n}(q, p); 0), \tag{3.18}$$

the Liouville condition of phase space incompressibility expressed in discrete language. Thus for evolving densities forward in time we need to evolve phase space points backward in time.

From the Liouville equation it is clear that if the initial density was uniform then it would remain uniform. That is the *uniform density is an invariant density or measure for all area preserving maps.* However if the map was not chaotic there exist other smooth invariant densities, and in general any initial density (that is not uniform) does not march on toward the uniform density as time goes by. Put in more technical language, in the space of densities, the uniform one is not an attractor, and there is general no attractor. However for chaotic systems it is almost as good as an attractor, and we illustrate this, not only by making use of Dubya but also by the following analysis.

The Dubya picture shows how complicated the outcome of the Liouville equation could be if the underlying dynamics f is chaotic. It appears from the picture that Dubya is uniformly spread all over the phase space. It is as if a "macroscopic" equilibrium has been reached after an initial scrambling. It is widely believed that this is the way in which thermodynamic equilibrium is reached, due to the mixing behaviour of the underlying dynamics. The evolution of densities such as Dubya's is in general difficult to track, as indeed it is tantamount to solving the equations of motion. However for the cat map this is possible, despite the presence of hard chaos. Since any density on the torus is doubly periodic we can write:

$$\rho(q, p; 0) = \sum_{kl} a_{k_1 k_2}(0)\, \exp\left(2\pi i k^T v\right), \tag{3.19}$$

where we have introduced the integer vector $k \equiv (k_1, k_2)^T$ and phase space $v \equiv (q, p)^T$. Here T in the superscript stands for the transpose. We note that from normalization of the density $a_{00}(0) = 1$. For a

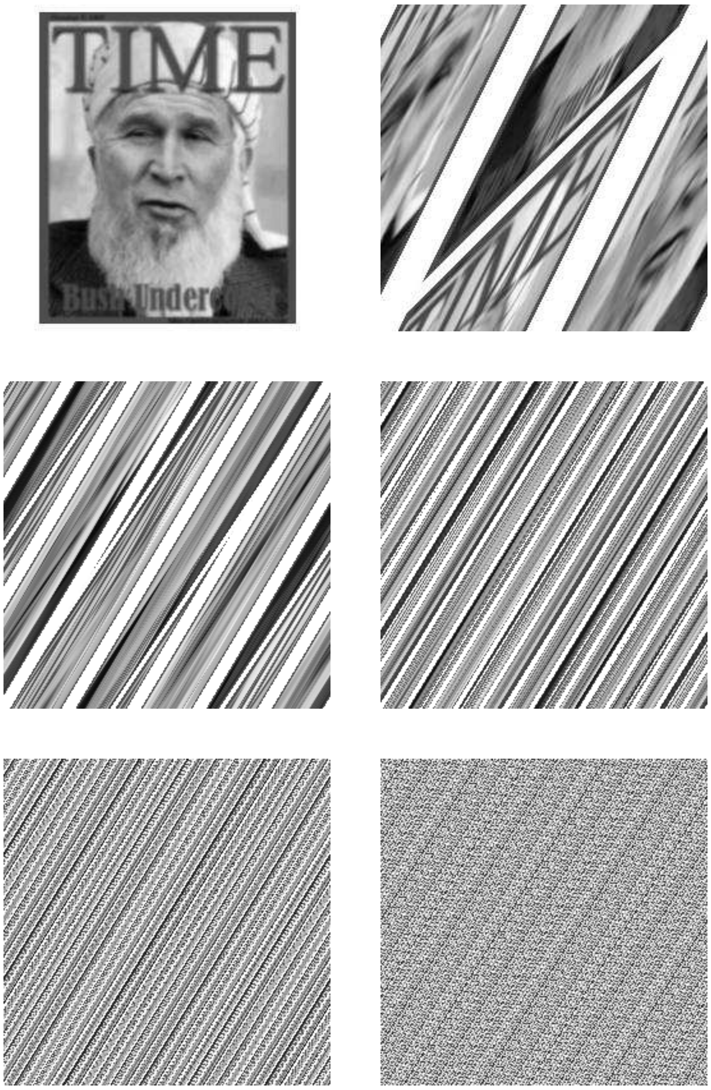

Figure 3.4: Dubya subject to the Arnold cat map for five iterations. Notice the unstable manifolds along which the pixels align, and how degeneration occurs with time. It is a peculiarity of the cat map though that there occurs a time when the initial image will exactly reappear, history will repeat endlessly, and perhaps needlessly. There is no loss of information in this process, the picture is merely scrambled. The original picture is taken without permission from http://politicalhumor.about.com/, where I highly recommend a visit to understand modern chaos in practice.

typical smooth function ρ the Fourier coefficients $a_{k_1 k_2}(0)$ will decrease with wavenumbers sufficiently fast for the series to converge.

The solution of the cat map is: $v_n = M^n\, v \,(\mathrm{mod}\ 1)$, where M is the 2×2 matrix $\{(a,b),(c,d)\}$. Note that this is only possible because M is an *integer* matrix, otherwise the act of taking powers of M do not commute with the modulo operation: the fact that the cat is a "diffeomorphism" is important here. Thus we have from Liouville equation Eq. (3.18) that:

$$\rho(q,p;n) = \sum_{kl} a_{k_1 k_2}(0)\, \exp\left(2\pi i k^T M^{-n} v\right) = \sum_{kl} a_{k_1 k_2}(n)\, \exp\left(2\pi i k^T v\right), \tag{3.20}$$

where $a_{k_1 k_2}(n) = a_{M^n k}(0)$. From the fact that M has real eigenvalues, one larger than 1, implies that $||M^n k||$ increases with time n for chaotic maps, for all k except $(0,0)$. (We have assumed for simplicity that M is symmetric, this is not of course essential). That is $a_{00}(n) = a_{00}$ for all n.

What we have done is transfer the cat map dynamics to a map on the "dual" Fourier space wavenumber lattice, on which the dynamics is a simple linear hyperbolic map. Note in particular that there is no "modulo 1" in this map. Therefore the small wavenumber coefficients get rapidly destroyed, as some higher wavenumber ones get "excited". In fact all this occurs exponentially rapidly. Thus the smooth density with predominantly small wavenumbers tends to be a highly oscillatory one, except for $a_{00} = 1$ for all time. Thus it is *not strictly true that the density will tend to the uniform density* ($\rho = 1$), however, it does so in a *weak* sense. that is

$$\int_\Omega f(q,p)\, \rho(q,p;n)\, dq\, dp \longrightarrow \int_\Omega f(q,p)\, dq\, dp. \tag{3.21}$$

as $n \to \infty$, for sufficiently smooth functions of phase space $f(q,p)$.

Thus the approach to equilibrium in Hamiltonian systems proceed in somewhat subtle ways, the complex network that Poincaré talked about, the homoclinic tangle leaves its imprint on the evolving density by means of high wavenumber features that would get observationally blurred, leaving behind an appearance of a smooth uniform, "microcanonical" ensemble.

3.1.2 Note on Cat-Mapping Any Image

You may have seen animated gifs or javas of Dubya-like pictures above. The problem is essentially one of the Liouville equations. For the animated GIFS shown in the lectures I used Mathematica to do this. Here I give the procedure, so that you can jump into it and try it for yourself, with different Cat maps and with different images. Those who are already computer savvy may please point out to me easier ways of doing this, also of course programs like Matlab may be used to do this. I assume that you are familiar with Mathematica.

Step 1. Have any image file in GIF or JPEG form. I prefer GIF so that we can animate it easily.

Step 2. Convert it into a black and white picture and into a square. You could do this for instance with GIMP. Say this file is called pic.gif.

Step 3. In a Mathematica session import this image using `g=Import["pic.gif"]`

Step 4. Get the pixel values of the image g, using `d=g[[1,1]]`. Now `d` should be a square array, the size depending on what you did in Step 2 (say $N \times N$). You may check this with the command `Dimensions[d]`, which should return only a pair of numbers, $\{N,N\}$ that should be the same, as the image is a square. If you see a third number that means you have a colour image and not BW. `d` is the square array of numbers that is your image. The cat map is a reversible mapping of this array of values. Imagine a $N \times N$ square grid on each grid point we have a pixel value, and then the grid points are mapped into other grid points taking along the pixels with them.

Step 5. In Mathematica do

```
nd = Table[d[[1 + Mod[i - j, N], 1 + Mod[-i + 2*j-1, N]]], {i, 1, N}, {j, 1,
       N}];
```

This is the crucial step. we have implemented the cat map $\{(2,1),(1,1)\}$, taking into account that the grid values go from 1 to N according to Mathematica and not 0 to $N-1$. Note that we implemented the *inverse* of the cat map, as applied to points. Why? `nd` is the transformed array.

Step 6. We can see this array with `Show[Graphics[Raster[nd/Max[nd]]], AspectRatio -> Automatic]`. Thats your image tranformed. Iterate this for additional steps, by using instead of `d`, `nd`.

Step 7. If you wish to create GIF animations. Then export the output of step 7 to a GIF image. Collect all such images you wish to animate. Say you have 10 of these. Call them pic.gif.000, pic.gif.001,...pic.gif.009. Then use any animation commands. I used "gifsicle", a simple command line program. I used the Linux version, there are many animators available. Mathematica itself will animate, but I dont think it can output an animated GIF. gifsicle can be found at http://www.lcdf.org/gifsicle/.

Step 8. Use your own map. Note that it has to take an integer to another integer. Have fun.

3.1.3 What Ho?

These last few pages, there has been a short overview of Hamiltonian dynamics in relation to chaos. Note that we have not really gone beyond 2 DoF or 2D area preserving maps. There is much more to be said, and much more to be found! 4D symplectic maps that are of relevance to 3 DoF Hamiltonian systems are only now being explored. "Arnold diffusion" is a process that is found to occur in these systems that is absent in lower dimensions.