Lecture 4

4.1 $\hbar \neq 0$ Introduction of Quantum Mechanics

The founders of quantum mechanics were adepts at the classical. In many cases classical intuition guided them along, till they reluctantly were forced to give in to notions that were genuinely alien then. For instance Schrödinger was guided by the analogy between wave optics and geometric (ray) optics, and made use of the Hamilton Jacobi theory heavily. It was with great reluctance that he introduced the imaginary unit $\sqrt{-1} = i$ fundamentally into physics. When he first wrote the equation $H\psi = E\psi$ and passed this on to Lorentz, Lorentz said it couldn't be quite right as an equation with H, that is energy must have somewhere time in it. See our very first Hamiltonian equations of motion, the Hamiltonian was crucially involved in finding time derivatives. After Schrödinger went back to the drawing board, he apparently came up with $H\psi = i\hbar\,\partial\psi/\partial t$, and was very unhappy with it, as it involved i. After one year he resolved this by applying H *twice* to get the real equation $H^2\psi = -\hbar^2\partial^2\psi/\partial t^2$. Only to realize that this cannot be right in general as the Hamiltonian can after all contain time explicitly, such as in the 1.5 DoF Hamiltonians we have been discussing. Thus it seemed that with reluctance he accepted the now famous Schrödinger equation which is our starting point.

The quantization of chaotic systems must need no motivation, considering the centrality of Hamiltonians in physics, considering the fundamental nature of quantum mechanics, and the alleged ubiquitousness of chaos, even in Hamiltonian systems. For instance going beyond the simplest atom, the hydrogen atom, to the Helium atom at once forces us to consider a problem in quantum chaos, as the two electrons in the field of the protons in the nucleus, is equivalent to a restricted three-body problem that is known to be chaotic. Even in the simplest of atoms, the H atom, if a sufficiently strong magnetic field was applied the electronic motion when treated classically is chaotic. If time dependent fields are applied almost anything (except the harmonic oscillator) is chaotic. All of these situations have also been realized experimentally.

More recently, the effects of quantum chaos on transport has been studied through measurements of magneto-resistance in quantum dots. These are simple generalizations of the 1D wells that we study in a first quantum mechanics course, only now they are 2D and can have different shapes. The motion of the electrons inside these is like the motion of a ball on billiard tables, and this class of billiard problems have been at the center of many classical as well as quantum studies. Experimental work on such systems has been on for more than a decade now. Recent explorations of the possibility of a quantum computer, has also brought to the forefront issues of the effect of chaos on the functioning of many-body quantum computers. In particular the "entangling" power of quantum chaos is a current research topic. Entanglement which measure how nonseparable quantum systems are in a genuinely quantum way, is influenced by chaos in an apparently very nontrivial manner. Apart from all these "systems", the fundamental nature of quantized chaotic systems could lie at the foundations of quantum statistical mechanics. There is also a suggestion that decoherence, which is the entanglement of a system with its environment, is strongly influenced by chaos in the system and/or environment.

The area of quantum chaos (we will call it that for simplicity, a certain school lead by Sir Michael Berry calls it "quantum chaology") is dominated roughly by the following approaches:

1. **Phenomenology**: Numerical experiments abound and are central to our understanding, much like the classical case. This provides the basis of many theories and modelling efforts.

2. **Semiclassics**: The semiclassical theory of anything beyond 1D (no chaos!), involves nontrivial extensions of the WKB method. Gutzwiller's periodic orbit theory developed in the late 60's, early 70's and refined over the years forms an important tool in understanding quantum chaos. Another semiclassical tool that has found practical applications and helped intuitive understanding is the technique of semiclassical wavepacket propagation.

3. **Statistical Modelling**: Chaos implies ergodicity, mixing, and an invariant measure classically. Thereby we can calculate averages knowing the invariant density. Quantum mechanically it has been found for about twenty years now that the methods originally devised for "complex" many body nuclei were applicable to low-dimensional chaotic systems. These methods fall under the class of problems which consider the Hamiltonian to be specific random matrices that obey the broad symmetries of the Hamiltonian of the system, such as parity, time-reversal symmetry etc. Thus modelling using random matrix theory (RMT) has been found to be of great importance and use in quantum chaos. RMT was developed initially by Wigner, Dyson and Mehta.

4. **Experiments**: Experiments from many fields (atomic, nuclear, mesoscopic, quantum optics) deal with quantum chaos, due to the basic character of the investigations, after all we are only talking of the Schrödinger equation. While the earliest experiments involved multiphoton ionization of hydrogen atoms in Rydberg states subjected to microwave radiation, recent controlled experiments range from quantum dots to laser cooled sodium atoms in pulsed fields, and resonant tunneling across quantum wells in magnetic fields. There are another class of experiments that are truly "electromagnetic" than quantum, in the sense that they simulate the Schrödinger equation with 2D microwave cavities (in which case the Helmoltz equation for the electric field becomes identical to the Schrödinger equation for the wavefunction of a particle freely moving inside a billiard whose shape is that of the 2D cavity.)

We may also classify the "objects" or "systems" of quantum chaos, that have been studied over the years. These are the central systems and are not exhaustive. The field of quantum chaos started with the independent and simultaneous publication of two papers, one on the quantized standard map by a combination of American (Joseph Ford), Italian (G. Casati) and Russian (B. Chirikov, F. M. Izrailev) efforts, while the other was on general planar 2D maps by a combination of American (N. L. Balazs), French (A. Voros) and British (M. V. Berry and M. Tabor) efforts, in 1979. Thus in fact the earliest quantized chaotic systems where quantizations of 1.5 DoF systems, and area preserving abstract maps that we have looked at classically in the previous chapter. Thus we begin the classification with these.

1. **Quantum Maps**: These come in two flavours: (a) quantizations of 1.5 DoF systems, mostly of the "kicked" variety that we saw lead to explicit stroboscopic maps; (b) quantization of abstract dynamical systems, such as the baker map, and the cat map. Although in some cases (like the cat map) there have been mappings on to kicked systems. Almost all studies so far on this have been restricted to 2D area preserving mappings.

   Experimentally these have been studied using pulsed fields that provide a very good approximations to delta function impulses. One important phenomenon studied here is the dynamical localization of wavepackets. Experimental realizations of the baker map have been proposed using simple quantum gates.

2. **Billiards**: Billiards are variously shaped cavities inside which particles travel freely and suffer specular reflections at the boundary. Mainly, 2D billiards have been studied. Apart from the circular, elliptical and rectangular 2D cavities, which are integrable, others studied are either chaotic or what is called "psuedo integrable". This is an intriguing set of models with application to quantum dots, where experiments using various shaped billiards have revealed that the presenc or absence of chaos has effects on conductance in ballistic dots.

3. **Nonlinear Oscillators**: Nonlinear oscillators, again mostly two oscillators have been studied. The phenomena of localization has been found here as well. These are of great importance in the study of atoms in strong external fields. In suitable coordinates these problems can be reduced to chaotic nonlinear oscillators.

4. **The Riemann Zeta function**: This rather odd addition to the "systems" is the tantalizing connections between number theory, analysis and chaos. The celebrated, largely unresolved, Riemann hypothesis conjectures that the (nontrivial) zeros of the zeta function lie on the $Re(z) = 1/2$ line. The zeta function obeys certain functional equations that look (almost) like the Gutzwiller semiclassical formulas, and the zeros, when treated as energy eigenvalues, are distributed according to a signature RMT distribution that is obeyed by time-reversal violating system. The millions of zeros of the zeta function have been used for statistical analysis, as if they are from a quantized chaotic system. Also the extensive theory of the zeta function has been used to gain insights into the semiclassics of quantum chaos.

   A spirited hunt was on to find the Hamitonian whose eigenvalues are the nontrivial (imaginary parts of) zeros of the zeta function, with the clues that its classical limit is chaotic and that it was not time reversal invariant, that its periodic orbits had periods that were the prime numbers and actions that were the log of the primes! The hunt is still on, although there are some developments on certain other parallel lines.

5. **Scattering systems**: We mention these separately, as we have not really talked about scattering, but chaotic scattering is quite common and has been studied mostly with the help of billiards. Experiments have been done using 2D microwave cavities.

6. **Spin Chains**: These are not really well studied models. But potentially interesting problems for the future include many body systems, such as spin chains studied in condensed matter physics. The Hubbard model, the Ising model in transverse field etc. are largely unexplored areas where quantum chaos could be found.

7. **Quantum Chromodynamics**: There has been some work that indicates that fundamental theories such as QCD would have connections to quantum chaos. As may be imagined, since we have not really gone much beyond 2DoF, a field theory is quite far. However again evidence from RMT statistics implies that chaos is present in QCD. This should not really come as a surprise, the very first application and motivation for RMT came from excited states of heavy nuclei.

We will concentrate on quantum maps in the following, due to their simplicity and usefulness.

4.2 Quantum Maps: Generalities

Quantum maps are quantizations of classical maps. Classical 2D area preserving maps are *symplectic* or what is more commonly called *canonical* transformations. Thus quantum maps may be thought of as the quantum equivalents of canonical transformations. The task is, given a canonical quantization, to find its quantum equivalent. The quantum equivalent is an unitary operator, and it is not immediately clear whether this is possible. For the class of maps derived from kicked Hamiltonians we can derive the quantum map quite easily as will be shown below.

We will first deal with the broad class of canonical transformations (maps) that comes from kicked systems. Consider the Hamiltonian:

$$H \;=\; H_0 \,+\, V \sum_{n=-\infty}^{\infty} \delta(t/T - n). \tag{4.1}$$

H_0 and V are time independent operators. There is no need to assume anything about the number of DoF. The Schrödinger equation we want to solve is

$$i\hbar \frac{\partial}{\partial t}|\psi\rangle \;=\; H_0|\psi\rangle \,+\, V\delta_T(t)|\psi\rangle; \tag{4.2}$$

given an initial state $|\psi(0)\rangle$, we wish to solve for $|\psi(t)\rangle$. Recall that $\delta_T(t)$ is an abbreviation for the train of delta impulses. Recall that the classical map related phase space variables just after consecutive kicks. Therefore we may consider the quantum map to be that operator, say U, that connects states just after consecutive kicks. Due to the periodicity of the perturbation, this is independent of the kick number itself, and from elementary quantum mechanics this is an unitary operator. Thus if we write the state just after kick n as $|\psi(n^+)\rangle$ we are seeking that U such that

$$U\,|\psi(n^+)\rangle \;=\; |\psi(n+1^+)\rangle. \tag{4.3}$$

From just after kick n to just before kick $n+1$, the delta function term is effectively absent in Eq. (4.2). Thus we have:

$$|\psi(n+1^-)\rangle \;=\; \exp(-iH_0T/\hbar)\,|\psi(n^+)\rangle. \tag{4.4}$$

Now we have to deal with integration over a kick, during which time the wavefunction changes abruptly. Integrating the schrödinger equation over the infinitesimal time about the kick, eliminates the H_0 part of the Hamiltonian, or alternatively we could say that the kick dominates the Hamiltonian during this time. Therefore consider the general equation:

$$i\hbar \frac{\partial}{\partial t}|\psi\rangle \;=\; V\,g(t)|\psi\rangle, \tag{4.5}$$

where $g(t)$ is any function of time t. It is easy to verify that it is solved by

$$|\psi(t')\rangle \;=\; \exp\left(-\frac{iV}{\hbar}\int_t^{t'} g(t)\,dt\right)|\psi(t)\rangle, \tag{4.6}$$

this effectively being a (dynamic) rescaling of time. In our case $g(t) \;=\; \delta_T(t)$ and we get using $t' = n+1^+$, $t = n+1^-$,

$$|\psi(n+1^+)\rangle \;=\; \exp(-iVT/\hbar)\,|\psi(n+1^-)\rangle. \tag{4.7}$$

Combining this with Eq. (4.4) we get:

$$|\psi(n+1)\rangle \;=\; \exp(-iVT/\hbar)\, \exp(-iH_0T/\hbar)\, |\psi(n)\rangle, \tag{4.8}$$

where we have dropped all the + superscripts, all the states are now understood to be immediately after the stated times. Thus we have the quantum map to be the unitary operator:

$$U \;=\; \exp(-iVT/\hbar)\, \exp(-iH_0T/\hbar) \tag{4.9}$$

4.2.1 Comments

The form of the unitary operator, or quantum map, is as a product of two unitary operators, the first one is the "free" evolution corresponding to the Hamiltonian H_0, while the other is the kick. If $[H_0, V] = 0$, that is these two operators commute then the evolution operator is simply $\exp(-i(H_0 + V)T/\hbar)$, equivalent to a continuous flow. The most interesting, and general case is when this does not happen. If the system has 1DoF (so that the whole Hamiltonian has 1.5 DoF), then this can mean the difference between order and chaos. Without the kicks the 1D Hamiltonian is simply $H_0 + V$, there is no classical chaos, the unitary operator over a time T is also $\exp(-i(H_0 + V)T/\hbar)$. Thus for 1D quantum systems the difference between e^{A+B} and $e^A\, e^B$ could spell the difference between order and chaos.

It must be noted that but for the impulsive time dependence we would be hard put to write explicitly the quantum map. For general time dependences, fully numerical integration has to be resorted to to write the corresponding quantum map or the Floquet operator. Since the Hamiltonian is time dependent the usual description of stationary states and energy eigenvalues does not exist. However the quantum map, is stationary in the sense that it does not depend on the kick number, and is a special case of the Floquet operator for general time dependence.

Iterating the classical map is equivalently done in quantum maps as repeated multiplication by the operator U, thus:

$$|\psi(n)\rangle \;=\; U^n\, |\psi(0)\rangle. \tag{4.10}$$

The powers of an operator is essentially determined by its eigenvalues and vectors and hence the problem of time evolution is here also solved by an eigenvalue problem. The eigenangles of U take the role of the energies and the eigenvectors of U are the "stationary" states. Since U is an unitary operator its eigenvalues lie on the unit circle, the eigenvalue equation may be written as:

$$U|\phi_k\rangle \;=\; \exp(i\phi_k)|\phi_k\rangle. \tag{4.11}$$

where $0 \leq \phi_k < 2\pi$ are real numbers, the eigenangles, and also referred to as the "quasi-energies". The index k may be discrete, or continuous, as the eigenangles are discrete or continuous. In fact the nature of the spectrum of operators such as U even for 1D systems is notoriously difficult to predict, it may also be of a singular continuous kind, where the spectrum is a Cantor set, a fractal. These different kinds of spectra impact crucially properties of quantum transport.

Note that

$$|\psi(n)\rangle \;=\; U^n\, |\psi(0)\rangle \;=\; \sum_k \exp(i\phi_k\, n)|\phi_k\rangle\langle\phi_k|\psi(0)\rangle. \tag{4.12}$$

From the reality of the eigenangles it follows that quantum evolution is quasi-periodic, especially if the spectrum is discrete. If the spectrum is continuous this need not be the case.

We now quantize the maps that were displayed in the classical section 1.3.1.

4.3 Quantum maps on the plane and cylinder

Quantum Linear Map. The quantum map corresponding to the classical map in Eq. (**??**) is

$$U = \exp\left(-\frac{iT}{\hbar}\omega^2 q^2\right) \exp\left(-\frac{iT}{2\hbar}p^2\right), \tag{4.13}$$

where of course p and q are operators satisfying the Heisenberg commutation relation: $[q,p] = i\hbar$. The quantum linear map is completely solvable, just like the classical linear map. This can be seen from the fact that an application of the Campbell-Baker-Hausdorff (CBH) theorem will reduce the product of the two exponentials to a single exponential with an effective quadratic Hamiltonian. Therefore for cases where the classical map is filled with elliptic orbits, the quantum spectrum is identical to the Harmonic oscillators, the eigenangles are equispaced.

Quantum Standard Map. The quantum map corresponding to the classical map in Eq. (**??**) is

$$U = \exp\left(\frac{igT}{(2\pi)^2\hbar}\cos(2\pi q)\right) \exp\left(-\frac{iT}{2\hbar}p^2\right). \tag{4.14}$$

While the classical map was shown to depend on only one (scaled) parameter, $K = T\,g$, in the case of the quantum map we see that there are *two* effective parameters: $gT/\hbar$ and $T/\hbar$. Thus $\hbar$ enters in a crucial manner, and a scaled Planck constant determines many behaviours of the standard map. We will return to the standard map shortly, but note that we have not taken into account any modulo operation so far. As it exists the above would be the quantization of the standard map on the entire plane. The nature of the spectrum of the above operator (whether discrete, continuous, singular) has been the topic of much research and the there has been a mapping of this map to an tight-binding model of condensed matter physics. It has been at the center of many quantum chaos studies including atom-optics experiments.

Quantum Harper Map. It immediately follows from the above that a quantization of the Harper map without the modulo conditions is

$$U = \exp\left(-\frac{igT}{(2\pi\hbar}\cos(2\pi q)\right) \exp\left(-\frac{iT}{2\pi\hbar}\cos(2\pi p)\right). \tag{4.15}$$

4.3.1 Standard map on the cylinder

While both the standard and Harper maps were easy to study classically, at least in terms of their phase space trajectories, the quantum maps pose severe problems, they are infinite dimensional and there is no guarantee that the spectrum is discrete. The classical maps lived on cylinders and tori and thereby simplified matters considerably. For instance the kicked pendulum configuration space is the circle S^1 and not the line. Therefore we could quantize these maps on cylinders and tori. Cylindrical phase spaces we are more familiar with. There is an angle $0 \leq q < 1$, and an angular momentum $-\infty < p < \infty$ pair, If $|q\rangle$ are angle kets, the angular momentum kets are $|m\rangle$ such that $p|m\rangle = mh$

$$|m\rangle = \int_0^{2\pi} e^{2\pi imq}|q\rangle\, dq;\ m = 0, \pm1, \pm2\ldots, \tag{4.16}$$

the transformation function between momentum and angle is:

$$< q|m\rangle = e^{2\pi imq}. \tag{4.17}$$

Thus the standard map on the cylinder is given the infinite dimensional matrix

$$U_{mm'} = \langle m| \exp\left(\frac{igT}{(2\pi)^2\hbar}\cos(2\pi q)\right) \exp\left(-\frac{iT}{2\hbar}p^2\right)|m'\rangle. \tag{4.18}$$

Inserting complete $|q\rangle$ states, we get

$$e^{-i2\pi^2 Tm'^2\hbar)}\frac{1}{2\pi}\int_0^{2\pi} e^{igT\cos(\theta)/(4\pi^2\hbar)}e^{i\theta(m'-m)}\,d\theta. \tag{4.19}$$

We have made the substitution $2\pi q = \theta$ and used the eigenvalues of the momentum operator $(2\pi m\hbar)$. Using the standard definition of the Bessel (J) function:

$$J_n(z) = \frac{1}{2\pi}\int_0^{2\pi} \exp\left(-i(n\theta - z\sin(\theta))\right)\,d\theta,$$

and the standard property of integer order Bessel functions: $J_{-n}(z) = (-1)^n J_n(z)$ we get:

$$U_{mm'} = (-i)^{m'-m} J_{m'-m}(gT/4\pi^2\hbar)\exp\left(-i2\pi^2 Tm^2\hbar\right), \tag{4.20}$$

$$= (-i)^{m'-m} J_{m'-m}(K/\tau\hbar)\exp\left(-i\tau m^2\hbar/2\right). \tag{4.21}$$

Here we have reintroduced the *classical* effective parameter $K = gT^2$ (see the classical map), and introduced $\tau = 4\pi^2 T$ to show that there is effectively one more parameter.

Note that we have already chosen our units so that the moment of inertia, I, of the rotator is unity, and the units of $\hbar$ are of I/τ, which may be considered a characteristic action in the system. Therefore one may define an effective scaled dimensionless Planck constant as $\hbar\tau/I = \hbar\tau$, as $I = 1$, which measures roughly "how many classical actions are there in an $\hbar$". Thus the classical limit is the limit $\hbar\tau \to 0$. Since always the scaled Planck constant appears in the combination $\hbar\tau$, we may set $\tau = 1$ and loosely talk of $\hbar \to 0$. Often one is bewildered how fundamental constants are blithely treated as variables and limits are taken, it is always implied in these cases that some scaled constant is involved. This can always be done as long as the fundamental constant is *not dimensionless.*

This matrix U is the quantum standard map on the cylinder. All the complications of the classical standard map, from the KAM theorem, the Poincaré Birkhoff theorem, to the homoclinic tangle and hard chaos, all illustrated in Fig. (**??**), must appear as $\hbar \to 0$ from this matrix, and must be in some quantized avatar otherwise. One of the first, and enduring, questions, of quantum chaos, was exactly how this happens. As M. V. Berry points out, there are two questions that are often conflated here: (a) How does the classical limit emerge from the quantum. The answer, if complicated, is already known in the sense that the classical limit is known. (b) Do any new phenomena occur in the semiclassical regime, something that is not classical, that perhaps survives from the quantum regime? Can this even survive the classical limit? As Berry again points out, this question would be nonsensical as long as the limit were not *singular.* The classical-quantum limit is completely different from say the relativistic-nonrelativistic limit in the sense that we cannot set $\hbar = 0$ and get the classical limit (as we may set $c = \infty$ and get the nonrelativistic limit).

The structure of the unitary operator U is that of a *banded matrix.* This follows from $|U_{mm'}| = |J_{m-m'}(K/\tau\hbar)|$. From properties of the Bessel function we know that $J_n(z)$ decays exponentially for $n > z$, thus the effective (half) band width of U is $K/\tau\hbar$. We see from this that in the classical limit the matrix tends to become full, so also in the large classical chaos limit when K is large. A full matrix has much more complicated spectra, at least potentially, than a banded one.

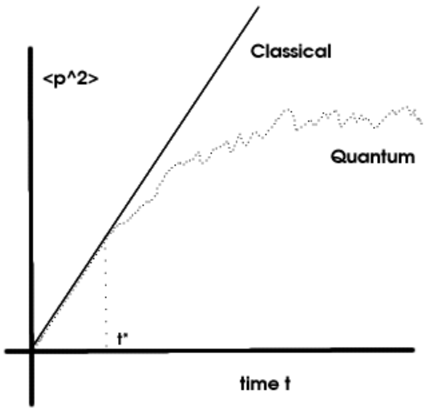

Figure 4.1: A doodle of the quantum and classical diffusion of the energy of the rotator.

One of the first questions asked of the quantum standard map, is whether it allows unlimited diffusion of the momentum of the pendulum. We know (see also "problem" in classical standard map section) that classically $< p^2(t) >= D\,t$, the kinetic energy, ensemble averaged over an initial conditions ensemble, increases linearly. Quantum mechanically an initial ensemble is already provided by a state, say take $|m = 0\rangle$ state, which is one of angular momentum zero. It is uniformly spread in the q space, and therefore we can think of this as the quantum equivalent of an ensemble of initial conditions with $p = 0$ and uniform spread in angle. Therefore we evaluate:

$$|\psi(t)\rangle = U^t|0\rangle;\ \langle p^2\rangle \ = \ \langle 0|U^{-t}p^2U^t|0\rangle \ = \ 4\pi^2\hbar^2\sum_m m^2|\langle 0|U^t|m\rangle|^2, \tag{4.22}$$

where in the last equality we have inserted momentum states. Shown below is a doodle of the quantum-classical cases. Note that this is an "easy" computation these days with powerful computers, however we have to truncate the infinite dimensional matrix somewhere, and this determines the time t till which our numerical calculation can be done. This is checked by ensuring that the norm of the vector $|\psi(t)\rangle$ is reasonably close to 1. A truncation leads to nonunitary matrices. The truncation becomes effective only if the matrix is banded and therefore from the above we see the difficulty posed by both the large chaos, and the semiclassical regimes.

The kinetic energy (actually twice the kinetic energy is shown in figure) of the quantum pendulum (or rotator) increases nearly linearly along with the classical till a time t^* after which it deviates, falls short and starts to oscillate. Whether this will continue to be bounded or will increase depends on the nature of the quantum spectrum and there are very few rigorous results on this. It is believed from many numerical investigations that the energy does not indefinitely increase. The fact that the energy of the rotator is bounded implies that all the eigenstates of the quantum map are *localized* in momentum space. This among things gave rise to the understanding that quantum mechanics somehow smoothens classical chaos, and makes it less random.

Problem.

1. *Establish that U is indeed an unitary matrix.*

2. *Write a program in your favourite language to calculate the matrix elements of U.*

3. *Find the powers of U by repeated multiplication and for the initial state $|m = 0\rangle$ find how the expectation value of the kinetic energy behaves in time. Note that one does not have to store the matrix elements if the elements are easy to find each time. Keep track of the normalization to see that it is not tending to 0. Suggested parameters are $\hbar\tau = 1$, $K = 5$, $N = 1000$, where N is the matrix dimension to which U is truncated. Note that for these parameters the band width is 5. if we go to a smaller $\hbar\tau$ like 0.01, the half width will be 500 and $N = 1000$ would be too small. Try it!*

The time t^* at which quantum-classical correspondence breaks down is an important time scale in much of quantum chaos. The general understanding is that it is at this time that the discreteness of the quantum spectrum is resolved. It is the so called Heisenberg time. There are more time scales in quantum chaos and a detailed study of these is quite involved and even incomplete. We do not attempt that here. It may be mentioned finally that such quantum localization of classical chaos was found experimentally by pulsing cold sodium atoms and the break time agreed well with theoretical predictions. Localization and delocalization remains a topic of research. One way the localization can be understood is by mapping the eigenvalue problem to a tight-binding model and show that the onsite potential is psuedo-random and invoke Anderson localization. The connections between transport properties in such disordered systems, and "simple" chaotic systems is intriguing and has been studied to some extent.