Lecture 5

5.1 Quantum maps on the torus

The quantum standard map on the cylinder is a scattering system, in the sense that unlimited diffusion in momentum can take place. If we want to study 2D Hamiltonian systems, the PSOS at any given energy in bound systems is finite, and therefore if we wish to study the effects of chaos, without the complication of scattering we have to study area preserving maps on compact phase spaces, such as tori. Indeed the two modulo conditions, on both q and p are possible for the standard map, the Harper map and is the natural phase space of the the baker and cat maps. This quantum maps on tori have been studied fruitfully as models of quantum 2 DoF nonintegrable Hamiltonian systems.

5.1.1 Quantum kinematics on the torus

Consider the unit $q-p$ torus $[0,1)\times[0,1)$ as the classical phase space. Since the restriction of both q and p to be angle like variables leads to finite quantum mechanics that is not often treated in standard books, we will develop this to the extent possible here. Imposing periodic boundary conditions on both position and momentum, results in a state space of finite dimensions. Let the position and momentum eigenstates be $|q_n\rangle$, and $|p_n\rangle$, $n = 0, 1, \ldots, N-1$, where N is the dimensionality of the Hilbert space. Assume that these form orthonormal basis sets. Recall from continuum quantum mechanics that $\exp(-ipa/\hbar)$ is the position translation operator in the sense that $\exp(-ipa/\hbar)|q\rangle = |q+a\rangle$,, while $\exp(iqb/\hbar)$ is the momentum translation (or boost) operator. Here a and b are arbitrary real numbers. Since there are only N discrete states, it is finite translation operators that take on special significance for finite quantum mechanics. Strictly speaking we should deal with unitary operators rather than the generating Hermitian ones, as there are no infinitesimal translations of a finite number of states. Therefore rather than deal with position and momentum operators we deal with position and momentum *translation* operators.

The position translation operator is denoted by T_q. It is such that

$$\langle q_n|T_q = \langle q_{n+1}|, \;\text{ and }\; \langle q_n|T_q^N = \langle q_{n+N}| = \exp(2\pi i\beta)\langle q_n|. \tag{5.1}$$

Here $0 \le \beta < 1$, is the phase accumulated as one goes along the q direction. For periodic boundary conditions $\beta = 0$, for anti-periodic, it is $\beta = 1/2$. Phases such as these are very useful in controlling quantum symmetries. The physical interpretation of this phase is possible as a magnetic flux line threading the particles circular configuration space. The phase then appears as the Aharanov-Bohm phase and is useful in controlling time-reversal symmetry. The translation operator in momentum, T_p, is similarly defined.

$$T_p|p_n\rangle = |p_{n+1}\rangle, \;\text{ and }\; T_p^N|p_n\rangle = |p_{n+N}\rangle = \exp(2\pi i\alpha)|p_n\rangle. \tag{5.2}$$

Here $0 \leq \alpha < 1$, similarly controls parity symmetry. We find the relationship between the two basis, the position and momentum, by requiring that the momentum states be eigenstates of the position translation operator and vice-versa; just as in continuum mechanics.

Exercises.

1. *Using the above definitions, prove that the translation operators T_p and T_q are indeed unitary. Write down the T_p operator in momentum basis and the T_q operator in position basis for $N = 4$.*
2. *Show by direct verification that*

$$|p_m\rangle \equiv \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \exp\left(2\pi i(n+\alpha)(m+\beta)/N\right)|q_n\rangle \tag{5.3}$$

is an eigenvector of T_q, that is

$$T_q|p_m\rangle = \exp\left(2\pi i(m+\beta)/N\right)|p_m\rangle. \tag{5.4}$$

Similarly

$$T_p|q_n\rangle = \exp\left(2\pi i(n+\alpha)/N\right)|q_n\rangle. \tag{5.5}$$

From this exercise we get the important transformation function which is nothing but a discrete version of the Fourier transform of continuum quantum mechanics.

$$\langle p_m|q_n\rangle = \frac{1}{\sqrt{N}} \exp\left(-2\pi i(n+\alpha)(m+\beta)/N\right) \equiv G_{mn} \tag{5.6}$$

Thus although we avoid writing down position and momentum operators, the rest of the machinery is very much there on the torus, and appropriate discretization is involved. We may also identify "position" and "momentum" eigenvalues from the eigenvalues of the translation operators as $q_n = (n+\alpha)/N$ and $p_m = (m+\beta)/N)$.

Notice that we have now a 2 parameter family of quantizations and this freedom with boundary conditions is absent in ordinary quantum mechanics. It is interesting that the Aharanov-Bohm phase appears in finite quantum mechanics unobtrusively and in fact there is one more phase which comes from momentum periodicity.

Exercise.

One may ask what about the uncertainty relation $[q,p] = i\hbar$? As is well known this translates into the fact that unitary translation operators along q and p, do not commute, which is a reflection that there is no sensible phase space in quantum mechanics. Show that the analogue of the uncertainty relation is the non-commutativity of T_q and T_p described by

$$T_pT_q = e^{-2\pi i/N} T_qT_p. \tag{5.7}$$

For simplicity put $\alpha = \beta = 0$. Hence find λ_{rs} where $T_p^rT_q^s = \lambda_{rs}T_q^sT_p^r$ for some integers r and s. Also investigate the role of the phases.

Where is $\hbar$? From comparing the transformation function with the continuum formula for $\langle q|p\rangle$ we may infer that $N = 1/h$. This also follows from semiclassical arguments of state counting: as there are N states on the unit phase space, and there is one state per h, it follows that

$$h = \text{Total area of phase space/Total number of states} = 1/N. \tag{5.8}$$

This is of course the effective scaled Planck constant. The classical limit is then the large N limit, $N \to \infty$.

Problem.

G_N, the finite Fourier transform matrix is an unitary matrix (prove it!). We discussed that quantum maps of area preserving transformations were unitary operators. May G_N be the quantum map corresponding to some classical map? Indeed show that G_N corresponds to the simple classical transformation of a 90 degree rotation of the unit square about its center. Note how this dovetails with the fact that $G^4 = 1_N$, where 1_N is the N dimensional identity. Hence show that the eigenvalues of G_N are only ± 1 and $\pm i$. Thus G_N, the finite Fourier transform has an interesting and intuitive interpretation of a rotation, a simple operation that is at the bottom of integrable systems.

Coherent States on the Torus

Coherent states on the torus are the equivalent of the coherent states that one studies in quantum mechanics as eigenstates of the harmonic oscillator annihilation operator. We use them essentially to visualize quantum states in a "phase space", as coherent states are of minimal $q - p$ uncertainty and therefore come closest to being phase space points. They are the quantum goggles through which we hope to see the quantum-classical border with more precision.

First we need a "fiducial" state which is the analogue of the harmonic oscillator ground state, the first coherent state (recall $a|0\rangle = 0$, so that $0\rangle$ is a coherent state). Saraceno proposed using the ground state of the Hamiltonian

$$H_h = 2 - \frac{1}{2}\left(T_q + T_q^\dagger \, + \, T_p + T_p^\dagger\right), \tag{5.9}$$

which is the quantization of $2 - \cos(2\pi q) - \cos(2\pi p)$ on the torus (see this from the analogy of the translation operators on the torus to the usual translation operators). Examine the classical Hamiltonian to see that the energy is bounded between 0 and 4, which is also the bound for the eigenvalues of H_h. Expanding about $q = p = 0$ we see that it is a Harmonic oscillator in the linear approximation. Thus we can expect this state to be localized at the $q = p = 0$ and be one of minimal uncertainty. Calling this state $|00\rangle$ we may then define:

$$|pq\rangle = T_p^p T_q^{-q} |00\rangle, \;\; 0 \le p, q \le N - 1, \tag{5.10}$$

so that $|pq\rangle$ is the coherent state that we seek. In particular if ψ is any state $|\langle pq|\psi\rangle|^2$ tells us how this state is spread in momentum and position simultaneously.

Problem.

Analyse more carefully the fiducial state as defined above. Derive an approximate expression for this from the harmonic oscillator approximation. Diagonalize H_h exactly and compare this ground state with that found approximately. What about the excite states? For how long do you expect the *harmonic oscillator approximation to be valid?*

This ends our discussion of the kinematics on the toral phase space. We are now ready to quantize maps on tori.

5.1.2 General toral quantum maps

Consider the quantization of the kicked Hamiltonian in Eq. (**??**) and the resultant classical map of Eq. (**??**). Recall that the map is from just after a kick to just after the next kick. Therefore first there is the "free motion", and then the kick. This results, as discussed earlier, in the quantum map:

$$U = \exp(-iV(q)/\hbar)\exp(-if(p)/\hbar). \tag{5.11}$$

We now wish to treat the map restricted to a torus, which requires that f and V be periodic functions of unit period. Thus we write the quantum map in the position basis as:

$$U_{nn'} = \langle q_n|U|q_{n'}\rangle = \langle q_n|\exp(-if(p)/\hbar)|q_n'\rangle e^{-2\pi i N V\left(\frac{n'+\alpha}{N}\right)}, \tag{5.12}$$

where we have used $h = 1/N$ and inserting complete momentum states we get

$$\langle q_n|\exp(-if(p)/\hbar)|q_n'\rangle = \frac{1}{N}\sum_{m=0}^{N-1} e^{-2\pi i N f(\frac{m+\beta}{N})} e^{2\pi i(m+\beta)(n-n')/N)}. \tag{5.13}$$

Finally the whole caboodle is:

$$U_{nn'} = \frac{1}{N} e^{-2\pi i N V\left(\frac{n'+\alpha}{N}\right)} \sum_{m=0}^{N-1} e^{-2\pi i N f(\frac{m+\beta}{N})} e^{2\pi i(m+\beta)(n-n')/N)}. \tag{5.14}$$

While this may appear somewhat messy, its essential underlying simplicity must be appreciated. Nothing much further can be done with this form in this generality. The finite unitary U is the quantum map corresponding to the general family of maps in Eq. (**??**), with modulo 1 conditions in both q and p. An example is the Harper map for which both f and V are simply cosines.

Exercise.

Write a program for the quantum Harper map. Evaluate the matrix elements and explicitly verify that it is unitary, that is $U^\dagger U = I_N$.

5.1.3 Quantum standard map on the torus

What about the standard map on the torus? In this case $f(p) = p^2/2$ does not appear naturally as a periodic function. However we define the action of the free motion on an momentum eigenstate in the obvious manner:

$$\exp(-ip^2/2\hbar)|p_m\rangle \ = \ \exp(-\pi i(m+\beta)^2/N)|p_m\rangle. \tag{5.15}$$

and therefore we quantize the standard map on the torus in the position basis as ($V = -K\cos(2\pi q)/(2\pi)^2$):

$$U_{nn'} = \frac{1}{N} e^{\frac{iNK}{2\pi}\cos\left(\frac{2\pi(n'+\alpha)}{N}\right)} \sum_{m=0}^{N-1} e^{-\pi i \frac{(m+\beta)^2}{N}} e^{2\pi i(m+\beta)(n-n')/N}. \tag{5.16}$$

In the case that $\beta = 0$, we can do the sum exactly, as this was done by Gauss quite some time back and constitutes an example of an exact *Gaussian sum.* The objection that p^2 is not periodic is addressed in this case by restricted the value of N to even integers. This ensures that $e^{-i\pi m^2/N}$ has the periodicity of N. We get after the summation a simple formula for the standard map on the torus:

$$U_{nn'} = \frac{1}{\sqrt{iN}} e^{\frac{iNK}{2\pi} \cos\left(\frac{2\pi(n'+\alpha)}{N}\right)} e^{i\pi(n'-n)^2/N}. \tag{5.17}$$

5.1.4 Quantum bakers map

The bakers map was the simplest classical 2D area preserving transformation that we studied, in the sense that it was exactly solvable for all orbit types, periodic or otherwise, its Lyapunov exponent, topological entropy etc. are easy to find, it is in short an exactly solvable model of chaos. Therefore it is natural for us to seek a quantum version of the map. Perhaps it will also be exactly solvable? It was with this motivation that the bakers map was first quantized in 1988-89, however, despite its structural simplicity it is not exactly solvable in the sense that we do not yet know how to solve for the spectrum of the quantum problem.

One reason why the baker was quantized fairly late was that it does not have a kicked Hamiltonian generating it. However there is a generating function generating it. Here we will merely state the quantum baker map and restrict ourselves to some motivating comments. Given a particular choice of toral boundary conditions (α, β), recall that we have the finite Fourier transform which we denoted by G_N as the transformation function between position and momentum. This will form the central quantity in the quantum baker map which in the position representation is:

$$B = G_N^{-1} \begin{pmatrix} G_{N/2} & 0 \\ 0 & G_{N/2} \end{pmatrix}. \tag{5.18}$$

Here $G_{N/2}$ is the $N/2$ dimensional Fourier transform (FT), and we require that N be an even integer. Thus we see that the quantum baker map is a simple combination of finite Fourier transforms. In terms of signals, it instructs us to split the signal into two halves and do a FT in each part individually, then combine the two outputs and do an inverse FT on the whole lot. Indeed it seems a very simple transformation, but this is the quantized version of the paradigm chaotic system: the left shift dynamics.

The bakers map first quantized by Balazs and Voros has since undergone several versions of justifications and other operators have also been proposed. We will take a route to it through generating functions that also throw interesting light on quantum-classical correspondences. As we had noted earlier, quantum maps are quantizations of classical canonical transformations.

Canonical Transformations and Propagators: A digression.

Recall that a canonical transformation is specified by a generating function and this comes in 4 flavors (Goldstein!), F_1, F_2, F_3 and F_4, such that

$$F_1(q,Q):\ p = \partial_q F_1,\ P = -\partial_Q F_1;\ F_2(q,P):\ p = \partial_q F_2,\ Q = \partial_P F_2; \tag{5.19}$$

$$F_3(p,Q):\ q = -\partial_p F_3,\ P = -\partial_Q F_3;\ F_4(p,P):\ q = -\partial_p F_4,\ Q = \partial_P F_4. \tag{5.20}$$

The generating functions F_2 and F_3 are said to be of a "mixed" kind, referring to their dependencies on the position and momentum. For the sake of simplicity we treat 1DoF and time independent transformations, the extensions do not add any essential complication.

Now we write the corresponding quantum propagators:

$$\langle Q|K|q\rangle \sim \exp(-iF_1(q,Q)/\hbar);\ \langle P|K|q\rangle \sim \exp(-iF_2(q,P)/\hbar); \tag{5.21}$$
$$\langle Q|K|p\rangle \sim \exp(-iF_3(p,Q)/\hbar);\ \langle P|K|p\rangle \sim \exp(-iF_4(p,P)/\hbar). \tag{5.22}$$

The propagators have 4 possible representations, two of which are mixed, and correspond exactly to the 4 classical generating function types. Notice that this has the intuitive structure of the propagator "taking the old variables to the new". The classical relation between the generating functions is via Legendre transformations (for instance $F_2 = F_1 + PQ$, F_1 is Legendre transformed wrt Q), while the quantum propagators are related to each other via the Fourier transform. The "$\sim$" symbol is to indicate that the relationship is semiclassical. However *if the transformation is linear the relationship is exact* Upton normalizations.

For instance take the identity transformation ($Q = q$, $P = p$), which is generated by $F_2 = qP$ (verify it), then since a classical identity transformation can be expected to be a quantum identity transformation we get $\langle P|Id|q\rangle = \langle P|q\rangle \sim \exp(-iPq/\hbar)$, a result we know is an equality if we put in the factor $1/\sqrt{2\pi\hbar}$. Thus the FT is the identity in the mixed representation. Consider the exchange of q and P, ($Q = p$, $P = -q$), which is actually a 90 degree anticlockwise rotation in phase space generated by $F_1 = qQ$. The corresponding quantum map is $\langle Q|K|q\rangle \sim \exp(-iqQ/\hbar)$, which is again the FT. Thus the FT is a 90 degree rotation in the pure position representation. This is also the solution to a problem posed earlier in the context of quantum maps. As another example, we know that Hamiltonians generate infinitesimal time translations, which leads us to $\langle P|K(dt)|q\rangle \sim \exp(-iH(q,P)\,dt/\hbar)$, while its Fourier transform would give us the propagator in position representation; but as mentioned earlier the FT is classically (and semiclassically) a Legendre transform. The Legendre transform of the Hamiltonian is of course the Lagrangian and we can write $\langle Q|K(dt)|q\rangle \sim \exp(-iL(q,Q)\,dt/\hbar)$, which is the mysterious formula in Dirac's book that spurred Feynman to develop the path-integral formalism of quantum mechanics. It also explains the confusion we often fall into while asking why the Lagrangian should appear in the propagator rather than the Hamiltonian, the answer is that the Hamiltonian would appear in a mixed representation of the propagator. That should be expected as indeed Hamiltonians generate translations in the "mixed space" of phase space.

We return to the baker map. Recall that the $L \to B$ part of the baker map is $(q' = 2\,q\,p' = p/2)$, which is generated by the canonical transform $F_2(q,P) = 2qP$, where we identify $(q' \equiv Q, p' \equiv P)$. Therefore this half is given by the operator B whose mixed representation is $\langle P|B|q\rangle \sim \exp(-i2qP/\hbar)$, which is almost the FT but for the factor 2 in the exponent. On the torus, this becomes $G_{N/2}$, see that this indeed has the right structure. However this is only for those ($q < 1/2$, $P < 1/2$), in the phase space. In the Hilbert space this translates to those ($q_n < 1/2$, $P_m < 1/2$) or ($n < N/2$, $m < N/2$). Similar arguments hold for the other half of the baker map and we get the quantum baker in the *mixed* representation by the matrix:

$$B_{\text{mixed (P-q) representation}} = \begin{pmatrix} G_{N/2} & 0 \\ 0 & G_{N/2} \end{pmatrix}, \tag{5.23}$$

from which we multiply by G_N^{-1} to take the baker to the pure position representation. This completes our motivation of why Eq. (5.18) should qualify as a quantum baker. Other arguments also favour this simple matrix. It is well known that quantization is not an unique process, and while there exist a few other quantizations, they are all equivalent (or believed to be) in the semiclassical regime. The simplicity of the Balazs-Voros baker makes it still the simplest model of quantum chaos. Saraceno found that antiperiodic boundary conditions ($\alpha = 1/2$, $\beta = 1/2$) maximally preserves the classical symmetry of reflection about the center of the square.

Exercise.

1. *Find explicitly the matrix elements of the quantum baker map in the position representation.*
2. *What is the quantum baker map in the momentum representation?*
3. *Define the operator R such that $R|q_n\rangle = |q_{N-n-1}\rangle$. Find the action of R on the momentum states. Hence prove that R corresponds classically to the act of reflection about the center of the square of the square: $(q \to 1-q,\, p \to 1-p)$. Show that the classical map commutes with this reflection.*
4. *Prove that $[R, B] = 0$ if $\alpha = 1/2, \beta = 1/2$.*
5. *Write a program to calculate the matrix elements of B and verify that indeed B is unitary and that it commutes with R.*