Lecture 6

6.1 Quantum Chaos: Eigenfunctions

We can use any of the models we have developed in the previous section, such as the quantum standard map or the quantum baker map to finally study the effects of quantizing a chaotic system. In particular if there are special properties of eigenfunctions and eigenvalues, and what happens on time evolution? We very briefly describe here some of the phenomenology found so far. As a practical note, we emphasize that none of the operators discussed thus far have been analytically diagonalized. Use any of the standard matrix diagonalization routines to diagonalize the matrices for some fixed N (effective inverse Planck constant) and parameter values when studying the stationary states.

6.1.1 Scarring.

Eigenfunctions of quantized chaotic systems are very interesting beasts that are full of information and importance. It is important to understand that quantum eigenfunctions are semiclassically associated with classically invariant sets. For classically chaotic systems, the only such set with full measure is the energy shell. There are smaller invariant sets, for instance there are an infinite number (dense, recall the baker map!) of periodic orbits whose measure is zero. It was not expected that these influence the eigenfunctions. This gave rise to the Berry-Voros hypothesis which said that the Wigner function (a phase space representation) of typical eigenstates will be uniformly spread on the energy shell (like the microcanonical ensemble of statistical physics). A mathematical formulation of this is known as Schnirelmann's theorem.

However numerical investigation of the eigenfunctions of the chaotic stadium billiard by Eric Heller showed that most of the eigenfunctions were "decorated" with strong streaks that could be unambiguously identified with classical periodic orbits. Since this is a chaotic system, these orbits are isolated and unstable. Heller called these "scars" of periodic orbits in quantum wavefunctions.

Since these investigations many many examples have been found. Heller has put forward certain explanations for this phenomena based on wavepacket dynamics, while Bogomolny and Berry have advanced semiclassical theories based on Gutzwiller's periodic orbit theory. Today we know that apart from periodic orbits there are also scars of homoclinic and heteroclinic orbits. However our understanding is far from complete, in the sense that we cannot say which eigenfunction will be influenced by which classical orbits. Heller gave a criteria that restricts those periodic orbits that could scar the wavefunctions, which effectively cuts off long periodic orbits. Almost all scarring studies have been done with 2DoF systems or quantum maps. Experimental evidence exists from resonant tunneling experiments and microwave cavity billiards.

It is accurate to say that eigenfunctions remain poorly understood even for 2DoF systems. When the baker map was first quantized, the bakers looked at the eigenfunctions in the position representation and concluded that they were mostly erratic, or random looking. However when subsequently coherent states were used to look at the functions a bewildering variety of patterns became visible, most of which could be identified with some classical feature, such as periodic orbits, or homoclinic orbits, or stable and unstable manifolds, but the way they were appearing held no particular rule. With the techniques developed by us so far, this is quite easy to do, we diagonalize the quantum baker map, we have used $N = 50$ in the figure below, find all the N eigenfunctions $|\psi_i\rangle$, which would be in the position basis and then calculate the overlap:

$$|\langle pq|\psi_i\rangle|^2, \tag{6.1}$$

for each i, where $|pq\rangle$ is the coherent state discussed earlier and must simply be thought of as a minimum uncertainty wavepacket at (q, p). The overlap is sometimes called the *Husimi representation.* This will tell us how the eigenfunction is spread in "phase-space."

Figs (6.1) are 30 of the 50 eigenfunctions and will give some idea of what we are up against. Firstly notice that the wavefunctions are not really spread all over, there is prominent clumping to be seen, and that this is indeed associated with classical orbits is quite easily seen in the case of the bakers map, as the classical orbits are exactly known. For instance we can see the period 2 orbit $(2/3, 1/3)$, $(1/3, 2/3)$ prominently in some eigenfunctions, we can also see period 3 and 4. To really convince onceselves, we need to superimpose the orbits with the eigenfunctions, and they would fall on each other. However in this "eigenfunction zoo" we only know know *aposteriori* what orbits are scarring what functions. Besides it must be said that there is a lot of "grass" or small structure that is not immediately visible but which is nevertheless important.

After the example of a "hard chaos" system, the baker map, lets look at the quantum standard map which could have a mixed phase space depending on the parameter K. Say we take $K = 2$, in which case we have a prominent elliptic island around the fixed point at the origin and a large chaotic sea. In Fig. (6.2) we again show 30 of the 50 eigenfunctions for $N = 50$ case. Firstly it appears overall "cleaner". It is useful now to go back to the classical standard map and look at the phase space for $K = 2$ in Fig. (**??**). See the remarkable classical-quantum correspondence that is there. Notice how there is a state that is only affected by the fixed point at the origin and how remarkably there is a state that is localized around the stable period 2 orbit $(0, 1/2), (1/2, 1/2)$. There are many that are affected by the elliptic orbits around the fixed point at the origin, and the rest are prominently located in the chaotic "sea" and look somewhat like the baker eigenfunctions. The other observation is that all the states seem to be either "scarred" by the regular orbits, or by the chaotic ones, not by both simultaneously. This was conjectured to be the case by Ian Percival in the early 1970's, and while it is seen to hold for many 2DoF systems, it is not clear if this is valid in higher dimensions. Violations of these have come to light in certain periodic systems, and states that are localized in both chaotic and regular regions have been called "amphibious".

A lot of work has gone into understanding functions, but there are many fundamental open questions concerning them. There is also a viewpoint that individual states may not be important, as there are too many and often they are too close in energy. Thus statistical and average properties may be of more experimental interest and indeed some finer semiclassical theories are possible if such averaging is done. We move on now to the eigenvalues.

6.2 Quantum Chaos: Semiclassics of Eigenvalues

The story of eigenvalues is somewhat more happier if not entirely so. The general aim of semiclassics is to provide quantum information from purely classical one. For instance everything about the quantum

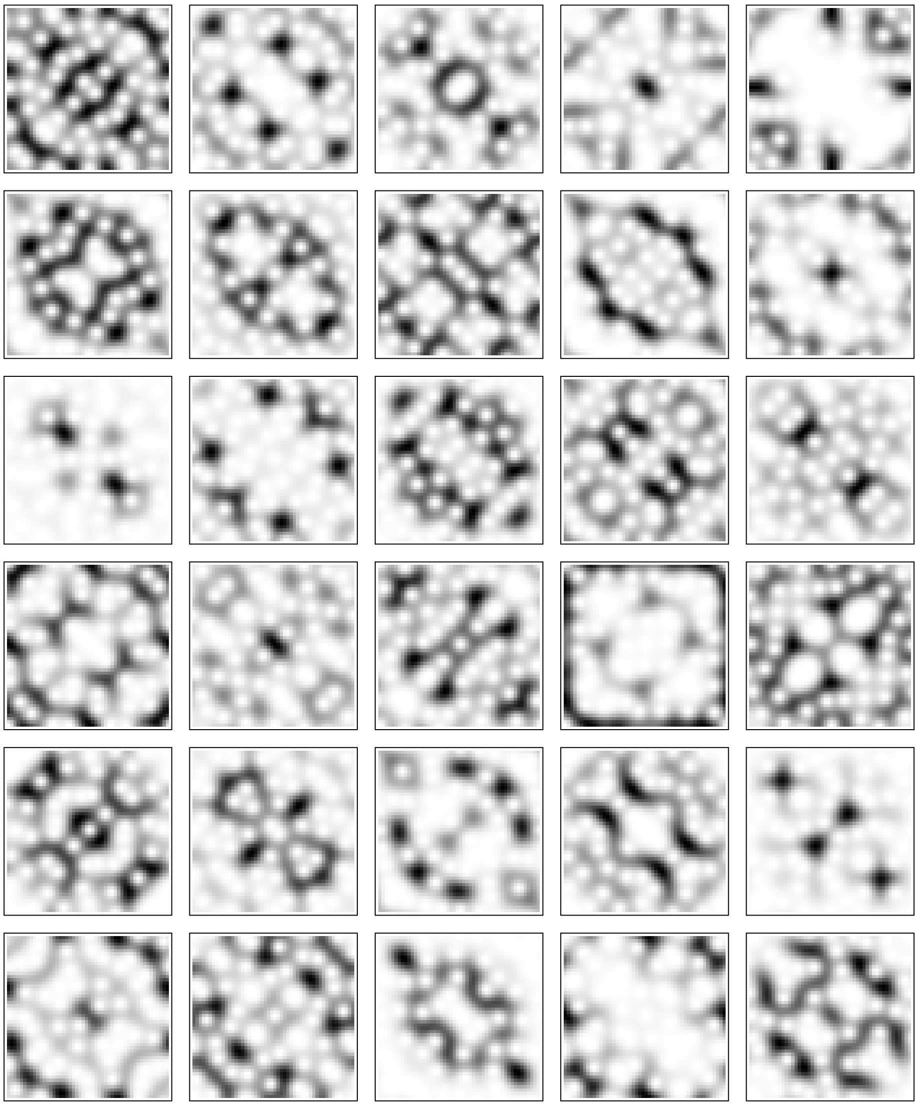

Figure 6.1: Quantum baker eigenfunctions for $N = 50$. The squares are the unit torus, notice intense scarring in some "clean" states by low period periodic orbits, 2, 3 and 4 orbit.

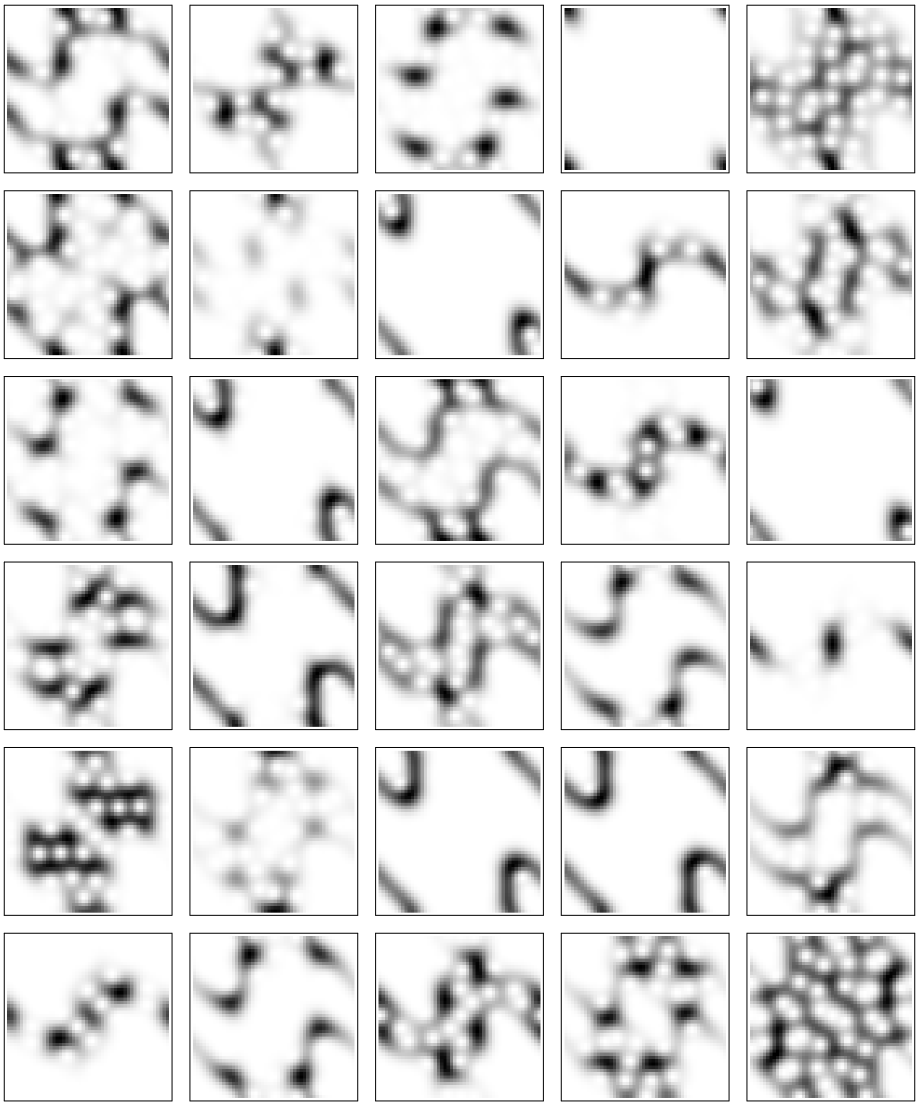

Figure 6.2: Quantum standard map eigenfunctions for $N = 50$. The squares are the unit torus, compare with the classical phase space in Fig. (**??**), and notice how all the eigenfunctions are either in the regular or in the chaotic regions.

harmonic oscillator can be found from classical properties. It is not at all clear whether this is possible in general. It must be emphasized that calculating quantum quantities using classical ones, does not mean that the phenomena are classical, of course not. It also does not mean that when quantum-classical correspondence there is an end to the use of semiclassics.

The Gutzwiller trace formula which connects the trace of the energy Green's function to a weighted sum over the periodic orbits on energy shells is the central object. This however requires quite an extensive treatment, we will here restrict ourselves to a part of it and give a flavor of the issues involved. Using quantum maps, the trace formula is also usefully studied. As one central object we will deal with the trace of the propagator in time:

$$\mathrm{Tr}(U^T) = \sum_{i=0}^{N-1} \exp(iT\phi_i) \tag{6.2}$$

where U is some unitary map, the ϕ_i are the eigenangles. This implies that if we know the traces of the powers of U we can know the eigenangles. This is seen by Fourier transforming both sides:

$$\sum_{T=-\infty}^{\infty} e^{-i\omega T}\mathrm{Tr}(U^T) = \tag{6.3}$$

$$\sum_{m=-\infty}^{\infty} \sum_{i=0}^{N-1} \delta(\phi_i - \omega - 2\pi m) \tag{6.4}$$

where we have made use of the *Poisson identity*:

$$\sum_{n=-\infty}^{\infty} e^{2\pi i n x} = \sum_{m=-\infty}^{\infty} \delta(x-m) \tag{6.5}$$

an identity from the theory of Fourier transforms that is of singular importance, simplicity and power. If we multiply the above identity by a smooth function $f(x)$ and integrate we get the following *Poisson summation formula*:

$$\sum_{n=-\infty}^{\infty} \int_{n=-\infty}^{\infty} f(x) e^{2\pi i n x}\, dx = \sum_{m=-\infty}^{\infty} f(x). \tag{6.6}$$

Thus Fourier transforming the traces of the powers of the propagator will give us delta functions at the eigenangles.

Semiclassical theory then attempts to find $\mathrm{Tr}(U^T)$ from purely classical quantities. The general structure of this trace formula is

$$\mathrm{Tr}(U^T) \sim \sum_{\gamma(T)} A_\gamma \exp(2\pi i S_\gamma/\hbar) \tag{6.7}$$

where γ enumerates classical *periodic orbits* of period T, S_γ is the action of the periodic orbit, and A_γ is a weight that typically exponentially falls with the period T. This last statement may give rise to the expectation that the long orbits can be neglected, but remember that in chaotic systems there is an *exponential proliferation* of periodic orbits with period. Therefore this is a delicate problem that needs care and in fact while the trace of quantum maps on finite Hilbert spaces hardly poses a problem, whole of semiclassics is beset with convergence issues. There is also the question of *how long* the above approximation holds good, while a fair amount of work has gone into this it is again not very clear what exactly this time is and how it depends on the scaled Planck constant and the parameters of chaos. For instance one surprising result that has emerged is that *semiclassics can be more accurate for larger chaos!*.

Semiclassics has been described as a determined usage of the stationary phase approximation. To give an idea of what goes on, we can make use of the quantum map machinery we have already discussed. As a fair warning this may be somewhat heavy on first reading, but this is really "semiclassics-lite".

Consider the following map defined on the unit torus:

$$\begin{aligned} q_{j+1} &= q_j + p_j - l_j \\ p_{j+1} &= p_j - V'(q_{j+1}) - m_j, \end{aligned} \tag{6.8}$$

where $V(q)$ is the periodic "kicking" potential with unit periodicity, the prime indicates derivative, and the integers m_j and l_j are such that the map is restricted to the unit torus (winding numbers). They are the "modulo one" operations. As was discussed earlier the above map can be derived from a time dependent Hamiltonian system. Also there exists a generating function for the map on the torus given by

$$S(q_j, q_{j+1}; l_j, m_j) = \frac{1}{2}(q_{j+1} - q_j + l_j)^2 - V(q_{j+1}) - m_j q_{j+1} \tag{6.9}$$

from which the map can be derived as $p_j = -\partial S/\partial q_j$ and $p_{j+1} = \partial S/\partial q_{j+1}$. This is of the F_1 type we discussed earlier.

The quantum map corresponding to the Eqs.(26) is given by the finite dimensional unitary matrix

$$\langle n|U|n'\rangle = \frac{e^{-i\pi/4}}{\sqrt{N}} \exp(\frac{i\pi}{N}(n-n')^2 - 2\pi i N V(\frac{n}{N})), \tag{6.10}$$

which generalizes the quantum standard map used before. This takes on the function of the propagator as it connects states after consecutive kicks. Here the representation is in the position basis and n and n' may take the integer values from 0 to $N-1$. N is itself the dimensionality of the Hilbert space and is also the inverse Planck constant, and is restricted to be even integers for preservation of toral boundary conditions. Semiclassics means the study of the unitary operator as $N \to \infty$.

The object of primary interest, the trace of the powers of the propagator, from which the spectrum may be derived by a Fourier transform can be written as

$$\mathrm{Tr}(U^T) = \frac{e^{-i\pi T/4}}{N^{T/2}} \sum_{n_i=0}^{N-1} \exp\left(\frac{2\pi i}{N} \sum_{j=1}^{T} \left(n_j^2 - n_j n_{j+1} - N^2 V(\frac{n_j}{N})\right)\right), \tag{6.11}$$

with

$$n_{T+1} = n_1.$$

It is implied in the above that the outer sum is over the T variables n_i, $i = 1, 2, \ldots, T$. Using the Poisson summation formula we can rewrite the above as

$$\begin{aligned} \mathrm{Tr}(U^T) &= e^{-i\pi T/4} N^{T/2} \sum_{k_i=-\infty}^{\infty} \int_{-\epsilon}^{1-\epsilon} dx_1 \ldots dx_T \\ \exp &\left(2\pi i N \sum_{j=1}^{T} \left(k_j x_j + x_j^2 - x_j x_{j+1} - V(x_j)\right)\right), \end{aligned} \tag{6.12}$$

with

$$x_{T+1} = x_1, \text{ and } k_{T+1} = k_1.$$

The stationary phase approximation, assuming that N is large, gives the following T conditions

$$x_{j+1} = 2x_j - x_{j-1} - V'(x_j) + k_j; \quad j = 1, \ldots, T, \; x_0 = x_T, \; x_1 = x_{T+1}, \tag{6.13}$$

with the further condition that

$$0 \leq x_j < 1.$$

These are precisely the equations that determine the period T orbits of the map. The integers k_j are related to the integers l_j and m_j of Eqs. (6.8) by the relation $k_j = l_{j-1} - l_j - m_{j-1}$. With the assumption that they uniquely determine a periodic orbit of period an integer fraction of T, we restrict the infinite sum to only those that relate to such a periodic orbit labeled below by γ. This is the first approximation and the second one is that we will extend the ranges of integration to the whole real line. These approximations together are exact in the case of the quantum cat maps, which we have not discussed here, and the third related approximation is that we Taylor expand the potential about the periodic orbits retaining up to the quadratic terms. This approximation is unnecessary for the piecewise linear maps, but is necessary in general.

Therefore we define new variables y_j such that $x_j = q_j + y_j$ and q_j is the central periodic orbit.

$$\begin{aligned} \mathrm{Tr}(U^T) \sim e^{-i\pi T/4} N^{T/2} \sum_{\gamma} \exp\left(2\pi i N \sum_{j=1}^{T} \left(-q_j q_{j+1} + q_j^2 + q_j\, k_j - V(q_j)\right)\right) \times \\ \int_{-\infty}^{\infty} dy_1 \ldots dy_T \exp\left(2\pi i N \sum_{j=1}^{T} \left(y_j^2 - y_j y_{j-1} - y_j^2\, V''(q_j)/2\right)\right) \end{aligned} \tag{6.14}$$

The sum

$$\sum_{j=1}^{T} \left(-q_j q_{j+1} + q_j^2 + q_j\, k_j - V(q_j)\right) \tag{6.15}$$

can be identified with the action of the periodic orbit, as one gets

$$S_\gamma = \sum_{j=1}^{T} (q_j^2 - q_j q_{j+1} - V(q_j) + q_j k_j + l_j^2/2), \tag{6.16}$$

We can neglect the term $l_j^2/2$, from the Lagrangian; this is legitimate as the action occurs as $e^{2\pi i N S}$, and since N is an even integer these terms do not matter.

The integral is evaluated by standard methods.

$$\int_{-\infty}^{\infty} d^T y \, \exp(i\pi N y^t A_\gamma y) = \frac{e^{i\pi T/4}}{N^{T/2}} \frac{e^{-i\pi\nu_\gamma/2}}{\sqrt{|\mathrm{Det}(A_\gamma)|}}. \tag{6.17}$$

Here y denotes the T component vector $\{y_j\}$ and y^t is its transpose. The real symmetric matrix A_γ is given by

$$A_\gamma = \begin{pmatrix} 2 - V''(q_1) & -1 & 0 & .. & .. & -1 \\ -1 & 2 - V''(q_2) & -1 & 0 & .. & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & 0 & .. & .. & -1 & 2 - V''(q_T) \end{pmatrix}. \tag{6.18}$$

ν_γ is a "Maslov like" index, and is the number of negative eigenvalues of the matrix A_γ.

The nontrivial problem of evaluating the determinant is already solved and the results are known. The determinant can be related to the stability of the periodic orbit. If the residue of the periodic orbit, which is simply related to the trace of the stability matrix, is R_γ, then $R_\gamma = -\frac{1}{4}\mathrm{Det}(A_\gamma)$. or if we denote the stability matrix eigenvalues by λ_+^γ and λ_-^γ, $\mathrm{Det}(A_\gamma) = \lambda_+^\gamma + \lambda_-^\gamma - 2$.

Thus finally we can write the asymptotic periodic orbit sum, or the trace formula, for the toral maps as

$$\mathrm{Tr}(U^T) \sim \sum_\gamma \frac{\exp\left(2\pi i N S_\gamma - i\pi\nu_\gamma/2\right)}{\sqrt{|\lambda_+^{\gamma T} + \lambda_-^{\gamma T} - 2|}}. \tag{6.19}$$

The sum is over periodic orbits whose periods are integer fractions of T are labeled by γ. This is of the canonical Gutzwiller-Tabor form having in the exponent the *action* of the period T orbits, and the prefactor explicitly depending on the *stability* of these orbits. The piecewise linear maps such as the baker map, or the sawtooth map are such that the Maslov like phases are zero and the only type of periodic orbits are those of the direct hyperbolic kind. With this formula we can study the vast class of *mixed* systems, such as the well known standard map. It is then of interest to ask questions such as how long the semiclassical approximations are valid in such systems which one knows are generic. Not much is known on these issues due to difficulty of finding *classical* orbits.

A careful application of the above formula in the case of the standard map, and for the simplest possible trace, the time 1 trace, yields the following:

$$\mathrm{Tr}(U) = \frac{e^{-i\pi/4}}{\sqrt{N}} \sum_{n=0}^{N-1} \exp(-ikN\cos(2\pi n/N)) \sim \tag{6.20}$$

$$\frac{e^{-i\pi/4}}{\sqrt{2\pi}} \left[\frac{2\sin(Nk+\pi/4)}{\sqrt{k}} + \sum_{l=1}^{[k]} \frac{4\sin(\theta_N(l)+\pi/4)}{(k^2-l^2)^{1/4}} \right], \tag{6.21}$$

where

$$\theta_N(l) = N(\sqrt{k^2-l^2} + l\arcsin(l/k)). \tag{6.22}$$

We have defined $k \equiv K/(2\pi)$, taken all quantum boundary conditions to be 0, and [] denotes the integer value function. Notice how the quantum trace involves a sum containing N terms, which in the classical limit has infinite number of terms, while the semiclassical approximation has only $[k]$ terms, depending on the chaos parameter in the system! The sum is over all the fixed points of the standard map, there are $[k]$ of them. Thus semiclassics exposes a *duality* between classical periodic orbits and quantum traces, a duality that is deep and goes into modern areas of mathematical research.

Problem.

Verify if the semiclassical approximation for the trace of the standard map propagator, Eq. (6.21) is correct. Write a program that will do both the sums in this approximation, and evaluate the goodness of the approximation as a function k and N.

Semiclassical formulae such as these are an important analytical tool in understanding quantum chaotic systems. Poincarë had predicted that periodic orbits would be the way to penetrate the fortress of dynamics, and this has been borne out as the truth in the case of semiclassics.

6.3 Random Matrix Theory.

We will unfortunately not be able to go into details of this important aspect of quantum chaos. As has been said earlier, the statistical aspect of quantum chaos is well described by random matrix theory (RMT), and is widely used in many areas from nuclear to mesoscopic physics. These days RMT is being used in widely different areas such as time series analysis of EEG data (from the brain), analysis of weather patterns and study of fluid flows.

Wigner suggested that since complex many body Hamiltonians were hopelessly difficult to solve, we may as well assume that the Hamitonian matrix (in a generic representation) is a random matrix, that is a matrix whose elements are drawn from a distribution at random. The model that is most often used is when the distribution is a Gaussian and the matrix elements are independently drawn. This leads to the Gaussian ensembles. There are some fundamental symmetries that systems may obey despite all the complexity and chaos. For instance time-reversal (TR) symmetry is quite fundamental. What does this imply for the Hamiltonian matrix? It implies that given any arbitrary basis, we can find a related basis in which the Hamiltonian is real. Therefore TR symmetric Hamiltonians are generically real, while TR nonsymmetric Hamiltonians are in general complex.

We have till now talked of unitary matrices, maps rather than Hamiltonians. Therefore the appropriate ensembles are called circular, as the matrices are unitary. For the results we will recall, the most simplest, it does not make a difference.

6.3.1 Nearest neighbour spacing distribution. (NNS)

This is one of the earliest measures that was used and is quite simple. The question is how are the nearest neighbour energy differences distributed? Somewhat paradoxically, if the system were integrable the NNS is Poisson distributed, which will happen if the energy levels do not have any correlation between them. If the system is chaotic (in the classical limit) the distribution will be one of two types depending of if there is TR symmetry or no TR symmetry. These are *universal* distributions, independent of the system and once full chaos has been achieved seemingly also indifferent to the degree of chaos. First we have to normalize the energy differences so that we can compare different spectra. To do this we normalize the levels so that the mean energy spacing is unity.

In the case of the unitary operators such as quantum maps, the mean spacing is simply $2\pi/N$ as the eigenangles are uniformly distributed over $[0, 2\pi)$. Therefore define

$$s_i = \frac{N}{2\pi}(\theta_i - \theta_{i-1}), \tag{6.23}$$

and consider the set of normalized spacings $\{s_i, i = 1, \ldots, N-1\}$. If these are distributed according to the function $P(s)$, RMT says that for the Gaussian ensembles:

$$P(s) = \begin{cases} \exp(-s) \text{ Integrable.} \\ \\ \frac{\pi}{2} s \exp(-\pi s^2/4) \text{ Chaotic. TR symmetric.} \\ \\ \frac{32}{\pi^2} s^2 \exp(-4s^2/\pi) \text{ Chaotic. No TR symmetry.} \end{cases} \tag{6.24}$$

For mixed systems, distributions are found that are intermediate between the Poisson and the Wigner. Notice that small spacings are the rule in integrable systems, while for chaotic ones, the energy levels are very rarely found close together. The levels are said to *repel* each other in chaotic systems.

Note that there is a linear repultion for TR symmetric systems and a quadratic repulsion for non-TR symmetric ones.

Exercise.

1. *Diagonalizing the quantum baker map using any algorithm, find the eigenangles. Normalize them so that the mean spacing is unity. Verify if the angles are uniformly distributed. Find the spacing distribution. You may take $N = 100$ or above for somewhat decent statistics. It has been found that while there is level repulsion, the quantum baker map has number theoretic properties that are not understood and leads to deviations from RMT predicted distributions.*

2. *Do the same exercise for the quantum standard map on the torus. Notice how for small K, where it is nearly integrable, the NNS distribution will be Poissonian ($\exp(-s)$), while for $K >> 5$, where complete classical chaos sets in the distribution will be RMT like. Show that if $\beta = 0$ the TR symmetric distribution is obtained, while if $\beta \neq 0$, the non-TR distribution is obtained. This shows how the phases may be used to make or break important symmetries that affect the spectrum dramatically. In fact this is a way in which one detect very weak TR violating effects in nuclear physics!*

RMT has also very important things to say about the eigenfunctions, this provides the base from which scarring is seen as deviations. We will not be able to go into these just now.

6.4 Afterword

We have given a somewhat whirlwind tour of the sights in this land of Hamiltonian chaos, but obviously a tourist's perspective is quite different from the natives'. I would like to end with a few topics of research that I see going on in quantum aspects of Hamiltonian chaos:

1. **Tunneling and Chaos:** While tunneling is uniquely quantum mechanical, it has been found that chaos has significant impact on tunneling rates. Tunneling rates are seen to be significantly enhanced in the presence of intervening chaos. Here again maps have been extensively used for these studies. The questions are quite involved as they must involve eigenfunctions.

2. **Fidelity Decay, Loschmidt Echo:** If we propagate an initial state by some Hamitonian H and reverse it by a slightly different Hamiltonian H' for the same time, the final state is not going to be the same. The fidelity, $F(t) = |\langle\psi|\exp(iH't/\hbar)\exp(-iHt/\hbar)|\psi\rangle|^2$, is a measure of how much it differs and is of practical significance in many situations. It may be imagined that if H is chaotic classically the fidelity will be more sensitive to how different H' is from H. The exact way in which $F(t)$ decays has been one topic of current research.

3. **Quantum information, entanglement, quantum computers, chaos:** The way in which quantum chaos impacts on quantum entanglement is being studied by several groups. This has larger ramifications in quantum information theory. Others have also studied if quantum chaos will aid or hinder a potential quantum computer. No definitive answers yet.

4. **$\hbar \longrightarrow 0$ or Decoherence?** The classical limit is still not completely clear, with one major school giving decoherence a place of central importance. Decoherence is the interaction of a pristine

isolated system with a nebulous environment. The fact that there are strictly no closed systems in physics, implies that these influences could be very important. quantum chaotic systems have very short time of quantum-classical correspondence and therefore have been used in these studies.

5. **Mesoscopic physics, transport in quantum dots.** Quantum chaos shows up in tiny devices as characteristic conductance fluctuations. Many experimental and theoretical studies are underway in these aspects.

6. **Semiclassics and RMT.** While RMT works it is not at all clear why? Afterall the quantum baker map, or the quantum standard map are very simple and definite matrices, nothing random about them. So where is the random coming from? There have been successful derivations of RMT results from semiclassics such as the trace or more general formulas, however there are important unresolved issues in TR symmetric systems.

7. **Higher dimensional systems** That we have concentrated on 2D area preserving maps is because higher dimensions are hardly explored. Some work has begun on 4D symplectic mappings and many body systems, but it is very preliminary stages.

A careful study of this document is enough to start addressing many of these questions. Whether this will end is entirely up to you. Happy hunting!

Further References

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I have of course not even attempted any sembelance of a bibliography. But there is much available in these sources and on line. Look at the searchable physics archives: http://in.arxiv.org in nonlinear science and quantum physics.