A tutorial on fluid dynamics, used in 2018. Solutions are straightforward and given along with. Key concepts of the subject are brought out nicely in this.

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PH1010-2018 **Tutorial 9 (corrected version)**

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1. The fact that incompressible fluids need to satisfy the condition $\nabla \cdot \mathbf{v} = 0$ (which directly follows from the equation of continuity) puts restrictions on the form of the velocity field $\mathbf{v}(\mathbf{r})$. For example, consider flow of a liquid out of a small hole of circular cross-section **in a two dimensional plane**, for which $\mathbf{v}$ has to be purely radial, by symmetry. Show that, in this case,

$$\mathbf{v} \propto \frac{\hat{\mathbf{r}}}{r} \quad (d = 2)$$

Next, consider a similar problem in three dimensions. Here, fluid is gushing out of a "point source" and spreading out radially. Show that, in this case,

$$\mathbf{v} \propto \frac{\hat{\mathbf{r}}}{r^2} \quad (d = 3)$$

You may find it convenient to use the expression for divergence in cylindrical and spherical polar coordinates, respectively, but it is not necessary.

In both cases, compute the vorticity $\mathbf{\Omega} = \nabla \times \mathbf{v}$.

2. Two-dimensional (or planar) flows of an incompressible fluid are often conveniently characterised by the stream function $\psi(x, y)$, such that $v_x = \partial\psi/\partial y$ and $v_y = -\partial\psi/\partial x$ (note the signs). In this case, the function $\psi(x, y)$ is constant along a given streamline. Graphically depict the velocity fields corresponding to stream functions (a) $\psi(x, y) = x^2 + y^2$ and (b) $\psi(x, y) = xy$. Try to imagine the physical situations where such flows are realised.

3. Consider an infinitely long cylinder of radius R and axis coinciding with the z-axis, rotating with angular velocity $\boldsymbol{\omega} = \omega\hat{\mathbf{k}}$ and immersed in a fluid. Assume that there is sufficient friction between the cylinder and the fluid such that the layer of fluid in contact with the surface of the cylinder is forced to move along with it. Assume that streamlines of fluid motion are concentric circles.

   (a) Show that the velocity field of the fluid is given by $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ (recall case (a) in the above problem). (hint: at the surface the cylinder, the velocity of the fluid and the cylinder should be the same)

   (b) Determine the vorticity $\mathbf{\Omega} = \nabla \times \mathbf{v}$ of the velocity field in (a).

4. Consider a paddle-wheel with two perpendicular paddles/fins, which can freely rotate about an axis (say, z-axis), and placed in flowing water. Consider a few different velocity fields as shown below, where v_0, ω_0 are positive.

$$\begin{aligned}
&(\mathbf{a})\ \mathbf{v}_1(x,y) = (v_0 + \omega_0 \frac{y}{2})\hat{\mathbf{i}} \\
&(\mathbf{b})\ \mathbf{v}_2(x,y) = (v_0 + \omega_0 \frac{x}{2})\hat{\mathbf{j}} \\
&(\mathbf{c})\ \mathbf{v}_3(x,y) = \mathbf{v}_2 - \mathbf{v}_1
\end{aligned} \tag{1}$$

Assume that a tip of a paddle moves with the same velocity as that of the layer of water in contact with it. Neglect loss of energy due to viscous dissipation.

In each case, find the angular velocity of rotation of the paddle (complete with direction), using the relation derived in 3(b) above. (hint: note that the problem considered here is the "inverse" of what was considered in (3), but the relation between vorticity of the velocity field and the angular velocity of the rotating object may be expected to be similar under similar conditions. Of course, cylindrical symmetry holds only approximately for a paddle wheel.)

5. ★ (a) Compare the Reynolds numbers corresponding to the following situations (i) a 30 m long whale, swimming in water with a speed 10 m s^{-1} (ii) a 1μm long bacterium, swimming at 30μm s^{-1} and (iii) an Olympic swimmer of height 2 m sprinting at nearly 2 m s^{-1}.

   (b) A dramatic illustration of the importance of Reynolds number is provided by the "coasting distance", which is the maximum distance a swimmer will move once it shuts down the propulsion mechanism. Compare the coasting distance of the three swimmers above. In all cases, imagine the swimmer to be a sphere of the same radius as the length given (in biology, such simplifications are often called "spherical cow approximations", see the joke quoted below*), and assume Stokes' viscous drag formula to be valid. (Useful info: take $\eta_{\text{water}} \simeq 10^{-3} N s m^{-2}$).

   (c) How is coasting distance related to Reynolds number ? (you need to assume here that mass densities of the swimmer and water is the same).

* From Wikipedia, the free encyclopedia

A spherical cow is a humorous metaphor for highly simplified scientific models of complex real life phenomena. The implication is that theoretical physicists will often reduce a problem to the simplest form they can imagine in order to make calculations more feasible, even though such simplification may hinder the model's application to reality.

The phrase comes from a joke that spoofs the simplifying assumptions that are sometimes used in theoretical physics.

Milk production at a dairy farm was low, so the farmer wrote to the local university, asking for help from academia. A multidisciplinary team of professors was assembled, headed by a theoretical physicist, and two weeks of intensive on-site investigation took place. The scholars then returned to the university, notebooks crammed with data, where the task of writing the report was left to the team leader. Shortly thereafter the physicist returned to the farm, saying to the farmer, "I have the solution, but it works only in the case of spherical cows in a vacuum".

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See solutions from the next page.

PH1010-2018 TUTORIAL 9 SOLUTIONS

1. Assume $\vec{v}(\vec{r}) = f(r)\,\vec{r}$

In $d=2$, $\vec{r} = x\hat{i} + y\hat{j}$, $r = \sqrt{x^2+y^2}$, $\vec{v} = f(r)x\,\hat{i} + f(r)y\,\hat{j}$

$\therefore\ \nabla\cdot\vec{v} = 2f(r) + x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$; $\frac{\partial f}{\partial x} = f'(r)\frac{\partial r}{\partial x} = f'(r)\frac{x}{r}$

Similarly, $\frac{\partial f}{\partial y} = f'(r)\frac{y}{r}$

$\therefore\ \nabla\cdot\vec{v} = 0 \Rightarrow \frac{x^2+y^2}{r}\frac{df}{dr} = -2f(r)$

$\Rightarrow \frac{df}{f} = -2\frac{dr}{r} \Rightarrow f(r) = \frac{C}{r^2}$

Hence, $\vec{v} = C\frac{\vec{r}}{r^2} = C\frac{\hat{r}}{r}$ in $d=2$.

In $d=3$, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $r = \sqrt{x^2+y^2+z^2}$

Then, $\nabla\cdot\vec{v} = 3f(r) + \frac{df}{dr}r = 0$

$\Rightarrow \frac{df}{f} = -3\frac{dr}{r} \Rightarrow f(r) = \frac{C}{r^3}$

Hence, $\vec{v} = C\frac{\vec{r}}{r^3} = C\frac{\hat{r}}{r^2}$ in $d=3$

It is easy to prove that $\vec{\Omega} = \nabla\times\vec{v} = 0$ in both cases, hence the flow is irrotational, or potential flow. Since $\nabla\times\vec{v} = 0$, we may write $\vec{v} = \nabla\phi$, where $\phi(\vec{r})$ is called the velocity potential. In the present case, $\phi \propto \ln r$ in $d=2$, and $\phi \propto \frac{1}{r}$ in $d=3$.

Ⓘ

2. Stream function

Note that by defining ψ through the relations

$$v_x = \frac{\partial \psi}{\partial y}, \quad v_y = -\frac{\partial \psi}{\partial x}, \quad \vec{\nabla}\cdot\vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$ as

required. Further, we also note that

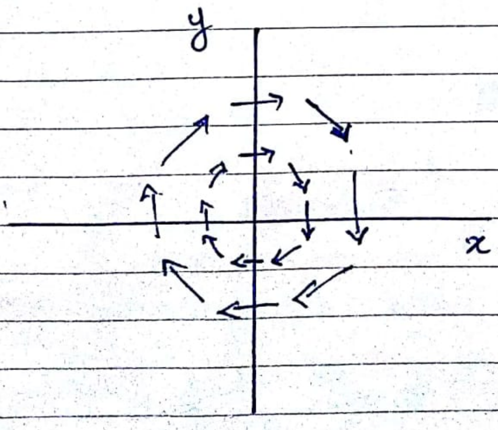
$$\vec{v}\cdot\vec{\nabla}\psi = v_x \frac{\partial \psi}{\partial x} + v_y \frac{\partial \psi}{\partial y} = 0$$ also, hence $\vec{\nabla}\psi$

is perpendicular to $\vec{v}$ at every point in space. Finally, since $\vec{v}$ is tangent to streamlines everywhere, $\vec{\nabla}\psi$ is normal to streamlines, hence ψ = constant along a particular streamline. Different streamlines, in general, correspond to different values of ψ.

(a) For $\psi(x,y) = x^2 + y^2$,

$$v_x = 2y, \quad v_y = -2x$$

$$\Rightarrow \vec{v} = 2(y\hat{i} - x\hat{j})$$



Since $\vec{v} \propto -\hat{\phi}$, and $|v| \propto \rho$, the fluid is rotating as a whole, with the axis of rotation being the z-axis.

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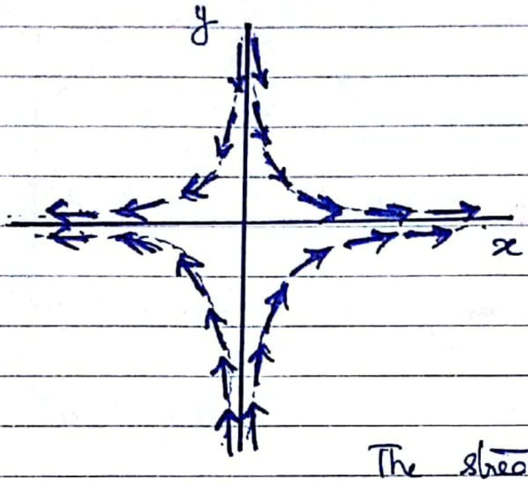
(b) For $\psi(x,y) = xy$, $v_x = \frac{\partial \psi}{\partial y} = x$, $v_y = \frac{-\partial \psi}{\partial x} = -y$

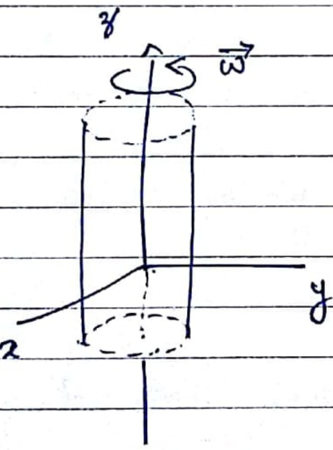
hence $\vec{v} = x\hat{i} - y\hat{j}$



In this case, each quadrant gives the fluid flow outside a wedge-shaped obstruction, the angle of wedge being 90°.

The streamlines here are hyperbolae.

3.



For streamlines being concentric circles outside the cylinder, let

$$\psi(x,y) = c\,(x^2 + y^2)$$

$$\Rightarrow \vec{v} = c\,[y\hat{i} - x\hat{j}] \quad \text{(see 2(a))}. \quad \text{—(1)}$$

On the surface of the cylinder, we need

$$\vec{v} = \vec{\omega} \times \vec{r} = \omega\,[-y\hat{i} + x\hat{j}] \quad \text{—(2)}$$

where $y = R\sin\phi$, $x = R\cos\phi$

~~By match~~

By matching the expressions (1) and (2) at the surface, we deduce that $c = -\omega$, hence

$\vec{v} = \vec{\omega} \times \vec{r}$ in the fluid.

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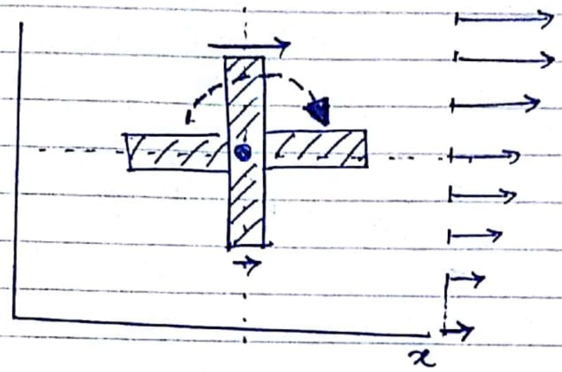
(b) $\vec{\Omega} = \nabla \times \vec{V} = \nabla \times (\vec{\omega} \times \vec{r}) = 2\vec{\omega}$ (see T8)

4.

(a)



The drag from the flow will cause the paddle-wheel to rotate clock-wise when seen from above, as indicated in the figure. The angular velocity of rotation is approximately given by

$$\vec{\omega} \approx \frac{1}{2}(\nabla \times \vec{V}) \quad \text{[see 3(b) above]}$$

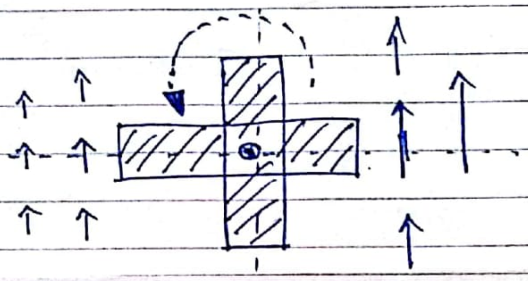
In this case,

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ V_0 + \frac{\omega_0 y}{2} & 0 & 0 \end{vmatrix} = -\frac{\omega_0}{2}\hat{k}$$

$$\Rightarrow \vec{\omega} = -\frac{\omega_0}{4}\hat{k}$$

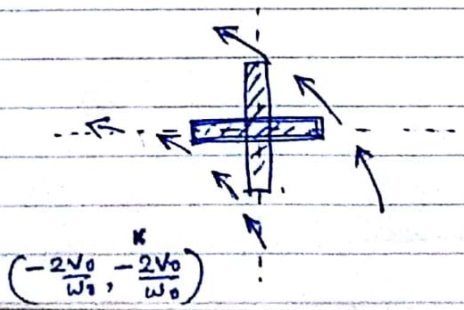
(V)

(b) In this case, following similar procedure as above, we find $\vec{\omega} \simeq \frac{\omega_0}{4}\hat{k}$. The rotation is counter clock-wise when seen from above.



(c) For $\vec{V} = \vec{V_2} - \vec{V_1}$, $v_x = -v_0 - \frac{\omega_0 y}{2}$, $v_y = v_0 + \frac{\omega_0 x}{2}$

Stream function $\psi(x,y) = -v_0(x+y) - \frac{\omega_0}{4}(x^2+y^2)$, hence streamlines are concentric circles with equation $\left(x + \frac{2v_0}{\omega_0}\right)^2 + \left(y + \frac{2v_0}{\omega_0}\right)^2 = \text{const}$.



$\Rightarrow$ CCW rotation of paddle.

Here, $\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -\left(v_0 + \frac{\omega_0 y}{2}\right) & \left(v_0 + \frac{\omega_0 x}{2}\right) & 0 \end{vmatrix}$

$\Rightarrow \vec{\Omega} = \omega_0 \hat{k}$, hence $\vec{\omega} \simeq \frac{\omega_0}{2}\hat{k}$ (CCW rotation)

VI

5. (a) Reynolds number $R = \frac{\rho u l}{\eta}$ where $\rho = \rho_{water} \simeq 10^3 \text{ kg/m}^3$.

(i) $R_{whale} = \frac{10^3 \times 10 \times 30}{10^{-3}} = 3\times10^8$

(ii) $R_{bacterium} = \frac{10^3 \times 30\times10^{-6} \times 10^{-6}}{10^{-3}} = 3\times10^{-5}$

(iii) $R_{human} = \frac{10^3 \times 2\times2}{10^{-3}} = 4\times10^6$.

(b) Assuming Stokes' drag law to be valid, Newton's 2nd law gives

$$m\frac{dv}{dt} = -6\pi\eta a v \quad \text{without propulsion}$$

$$\Rightarrow v = v_0 e^{-\frac{6\pi\eta a}{m}t} \quad V_0 = \text{initial velocity}$$

Coasting distance $d_{max} = \int_0^\infty v\,dt = \frac{V_0 m}{6\pi\eta a} = \frac{V_0 \times \rho \times \frac{4}{3}\pi a^3}{6\pi\eta a}$

$$\Rightarrow \frac{d_{max}}{a} = \frac{V_0 \rho a}{6\eta} = \frac{R}{6}$$

~~possibly~~ indicates a breakdown of Stokes' drag formula for large R

For whale, $d_{max} \sim 10^9$ m, while for human, $d_{max} \sim 10^6$ m, unrealistically large distances (but note that v decays exponentially fast). For bacterium, $d_{max} \sim 10^{-11}$ m, less than 1 Å!

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