The idea of circulation

from a summay of class notes of Prof. C. Vijayan

$$circulation = \oint_C \vec{A}(\vec{r}).d\vec{l}$$

We need a vector and a closed path to define circulation

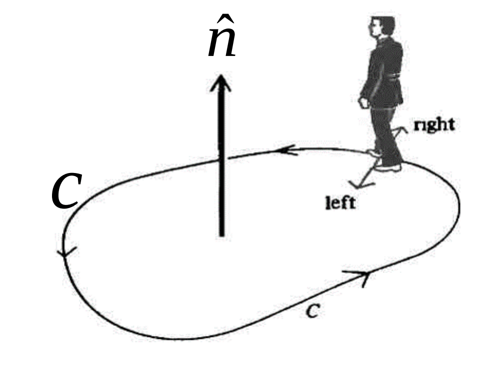
Find the dot product (of the vector and the line element) at every point on the curve and add up → integrate along the curve.

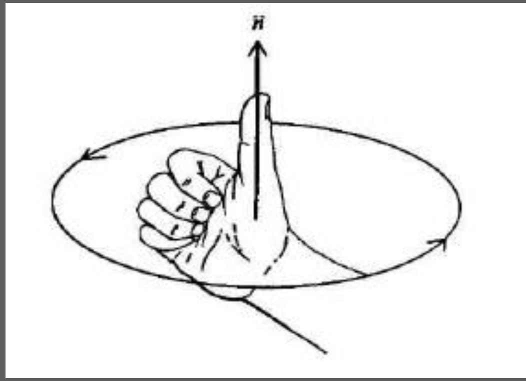
Consider an open surface *S*, bounded by a closed curve *C*. Circulation depends on the value of the vector at ***all*** points on C and hence is not a scalar *field*. ***i.e.***, NOT defined at a point, but only with respect to a curve

Consider an open surface *S*, bounded by a closed curve *C*. Shrink the curve C till, in the limit, the length of the curve tends to zero. Then the area of the surface the S bound by C also tends to be zero. But the ratio is nonzero and finite in the limit -- the 'curl' is a local quantity at that point.

$$(curl\ \vec{A}).\hat{n} = \lim_{\delta S \to 0} \frac{\oint_C \vec{A}.d\vec{l}}{\delta S}$$

The direction of the unit vector is normal to the area *S*   bound by the curve *C*





The physical meaning of curl

a paddle wheel placed at various points in a moving fluid would tend to rotate in regions where

$curl\ \vec{F} \neq \vec{0}$

The vector filed is irrotational if the curl is zero everywhere. E.g.: conservative force fields

- curl of a vector field at a point represents the net circulation of the field around that point.
- the magnitude of the curl vector at any point represents the maximum circulation at any point.
- the direction of the curl vector (use right-hand-rule) is normal to the surface upon which the circulation is the greatest.

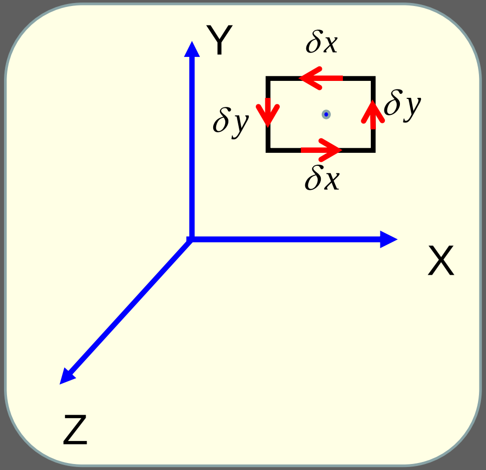
Remember! The criterion that a force field is conservative is that its path integral over a closed loop (i.e. "circulation") is zero; equivalently, its curl is zero.

Important result 1
curl of a gradient is zero

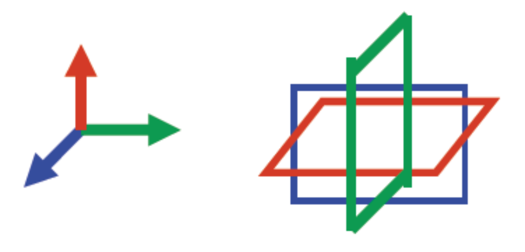
Important result 2
divergence of a curl is zero

Prove these two results.

Curl in Cartesian Co-ordinates



Consider the point (x_0, y_0, z_0)



Circulation for the loop in the xy-plane is

$$\left[A_x\left(x_0, y_0 - \frac{\delta y}{2}, z_0\right) - A_x\left(x_0, y_0 + \frac{\delta y}{2}, z_0\right)\right]\delta x + \left[A_y\left(x_0 + \frac{\delta x}{2}, y_0, z_0\right) - A_y\left(x_0 - \frac{\delta x}{2}, y_0, z_0\right)\right]\delta y$$

$$\oint_c \vec{A}(\vec{r}).d\vec{l} = -\frac{\partial A_x}{\partial y}\delta y \delta x + \frac{\partial A_y}{\partial x}\delta x \delta y$$

but, $$\left(curl\ \vec{A}\right) . \hat{e}_z = \frac{1}{\delta x \delta y}\oint_c \vec{A}(\vec{r}).d\vec{l}$$

$$\Rightarrow \left(curl\ \vec{A}\right) . \hat{e}_z = \left\{\left(curl\ \vec{A}\right)\right\}_z = -\frac{\partial A_x}{\partial y} + \frac{\partial A_y}{\partial x}$$

In a similar manner, considering the circulation along loops in the other faces,

$$curl\ \vec{A} = \hat{e}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{e}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{e}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\vec{\nabla} \times \vec{V} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

Remember! Curl is NOT a determinant !!!

This is just a 'memnonic' *i.e.* an way way to remember

The del operator (∇) is a vector operator

$$\vec{\nabla} \equiv \left[\hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial_z}{\partial z} \right]$$

When it operates on some scalar $f\,(x,y,z)$, we get **gradient**, a vector quantity.

$$\vec{\nabla} f\,(x,y,z) \equiv \left[\frac{\partial f}{\partial x} \hat{e}_x + \frac{\partial f}{\partial y} \hat{e}_y + \frac{\partial f}{\partial z} \hat{e}_z \right]$$

When we take the dot product of the del operator with some vector $\vec{F}(x,y,z)$, we get divergence, a scalar quantity.

$$div\ \vec{F} = \vec{\nabla}\ .\ \vec{F} = \left[\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right]$$

When we take the cross product of the del operator with some vector $\vec{F}(x,y,z)$, we get curl, a vector quantity.

$$curl\ \vec{F} = \hat{e}_x \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \hat{e}_y \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \hat{e}_z \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

Vector calculus in polar co-ordinate systems

$$\nabla u = \frac{\partial u}{\partial \rho}\hat{e}_\rho + \frac{1}{\rho}\frac{\partial u}{\partial \varphi}\hat{e}_\varphi + \frac{\partial u}{\partial z}\hat{e}_z$$

$$\nabla . \vec{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \vec{A} = \left[\frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right]\hat{e}_\rho + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right]\hat{e}_\varphi + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho A_\varphi) - \frac{\partial A_\rho}{\partial \varphi}\right]\hat{e}_z$$

$$\nabla u = \frac{\partial u}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial u}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial u}{\partial \varphi}\hat{e}_\varphi$$

$$\nabla . \vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial \varphi}$$

$$\nabla \times \vec{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta A_\varphi) - \frac{\partial A_\theta}{\partial \varphi}\right]\hat{e}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \varphi} - \frac{\partial}{\partial r}(r A_\varphi)\right]\hat{e}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial \theta}\right]\hat{e}_\varphi$$

https://www.geogebra.org/m/UstetaTp

Stokes' theorem

For a path δC, which binds an area δS,

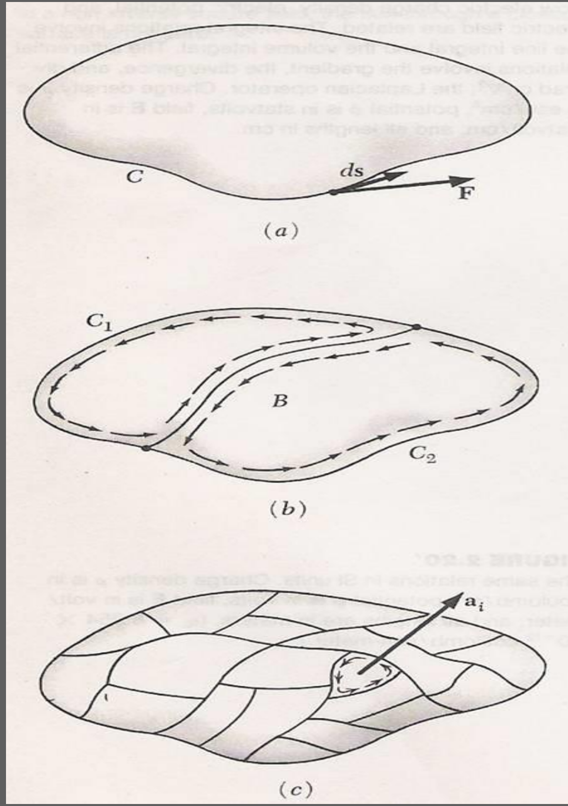
$$(curl\ \vec{A}).\hat{n} = \lim_{\delta S \to 0} \frac{\oint_C \vec{A}(\vec{r}).d\vec{l}}{\delta S} = (\vec{\nabla} \times \vec{A}).\hat{n}$$

$$\oint_{\delta C} \vec{A}.d\vec{l} = \delta S(curl\ \vec{A}).\hat{n}$$

A finite area S is enclosed by C, can be split up into infinitesimal bits δS_n bounded by curves δC_n

$$\oint_C \vec{A}.d\vec{l} = \sum_{i=1}^{n} \oint_{\delta C_i} \vec{A}.d\vec{l} = \sum_{i=1}^{n} \int_{\delta S_i} \{curl\ \vec{A} \cdot d\vec{S}\}$$

$$\oint_C \vec{A}.d\vec{l} = \int_S \left(\vec{\nabla} \times \vec{A}\right).d\vec{S}$$



Picture source: Electricity and Magnetism, Vol. II, Purcell (Berkeley Physics Course), McGraw-Hill, 1984

The Stokes' theorem relates the line integral of a vector about a closed curve to the surface integral of its curl over the enclosed area that the closed curve binds. Any area corresponding to C, bounded by C will work. (There could be many; see picture)
(only for orientable surfaces, not the Mobius type)

The same curve can correspond to several different areas

Examples for *curl*

- In a tornado the winds rotate about the eye and a velocity field would have a non-zero curl at the eye and possibly elsewhere.

  In a vector field that describes the linear velocities of each part of a rotating disk, the curl will have the same value on all parts of the disk.

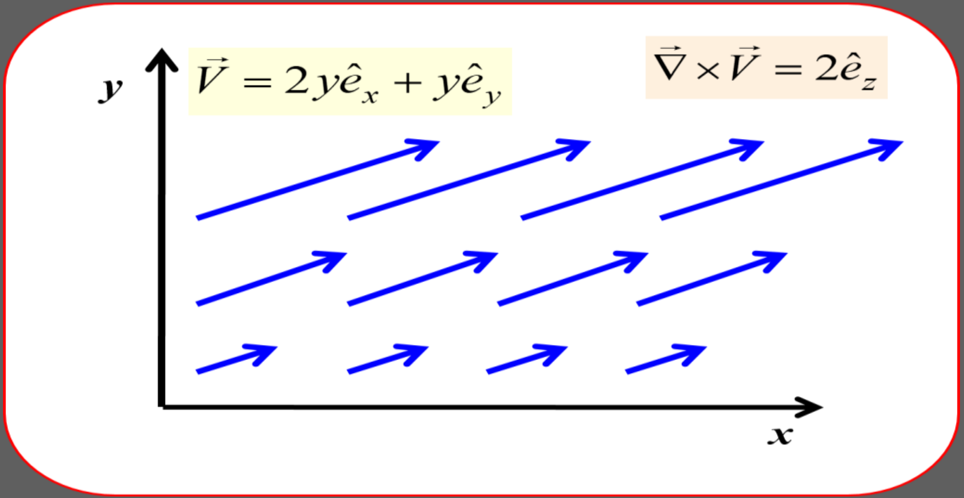
- If velocities of cars on a freeway were described by a vector field and the lanes had different speed limits, the curl on the borders between lanes would be non-zero.

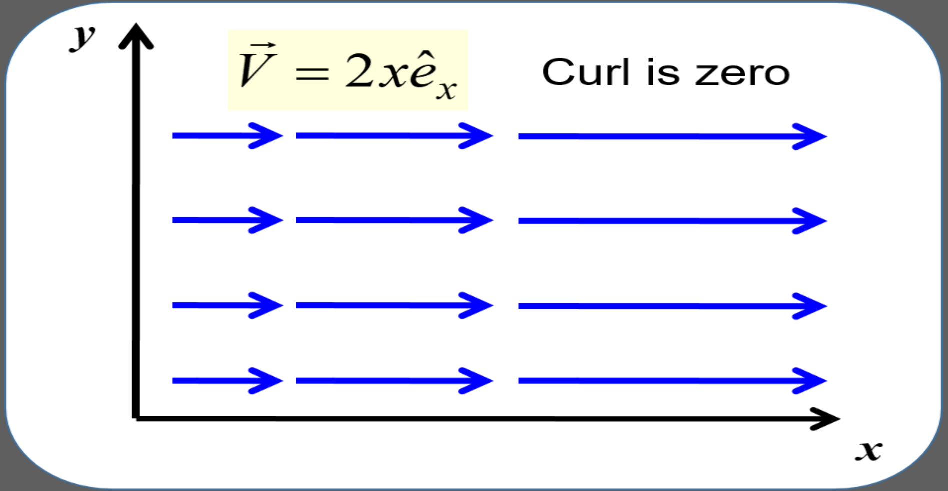
$$\vec{v} = \vec{\omega} \times \vec{r}$$

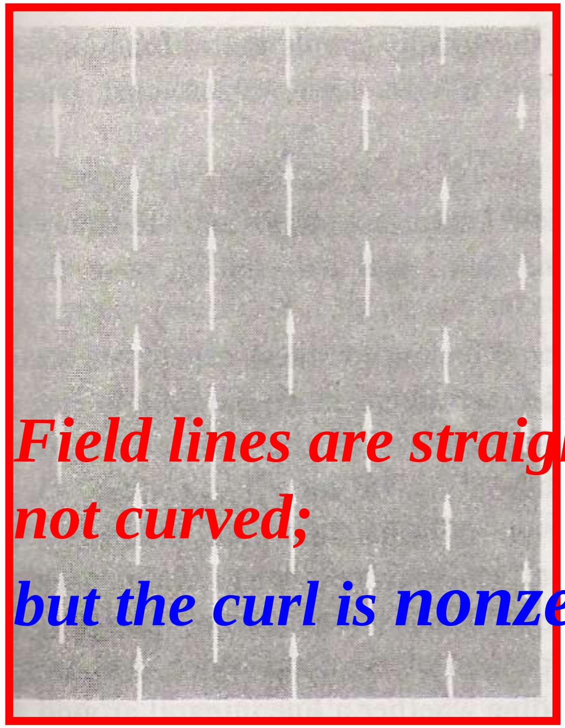
$$\vec{\nabla} \times \vec{v} = 2(\omega_1 \hat{e}_x + \omega_2 \hat{e}_y + \omega_3 \hat{e}_z) = 2\vec{\omega}$$

The 'curl' of the linear velocity gives you a measure of the angular velocity.

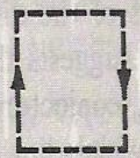
Picture credits:
https://kevinmehall.net/
A good site to play with





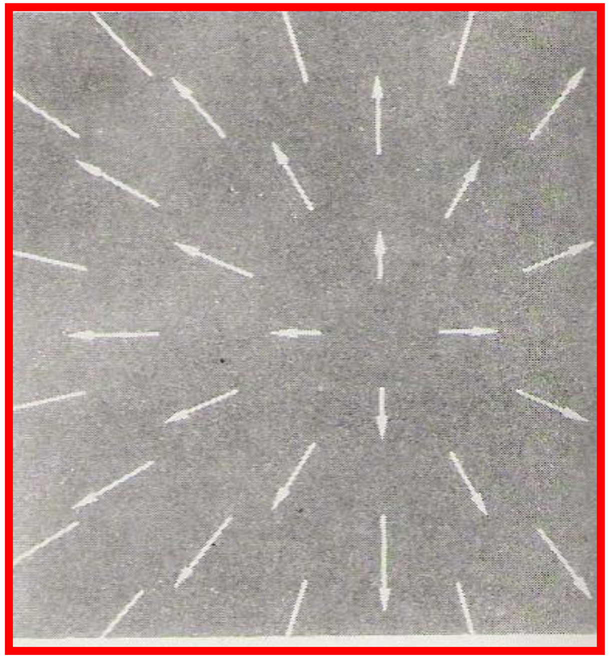


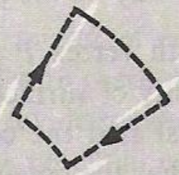
Note that the vector remains constant as you advance in the direction in which it points. That is, $\partial F/\partial y = 0$, with $F_x = 0$. Hence $\text{div}\,\mathbf{F} = 0$. Note that the line integral around the dotted path is not zero.



$\text{div}\,\mathbf{F} = 0 \qquad \text{curl}\,\mathbf{F} \neq 0$

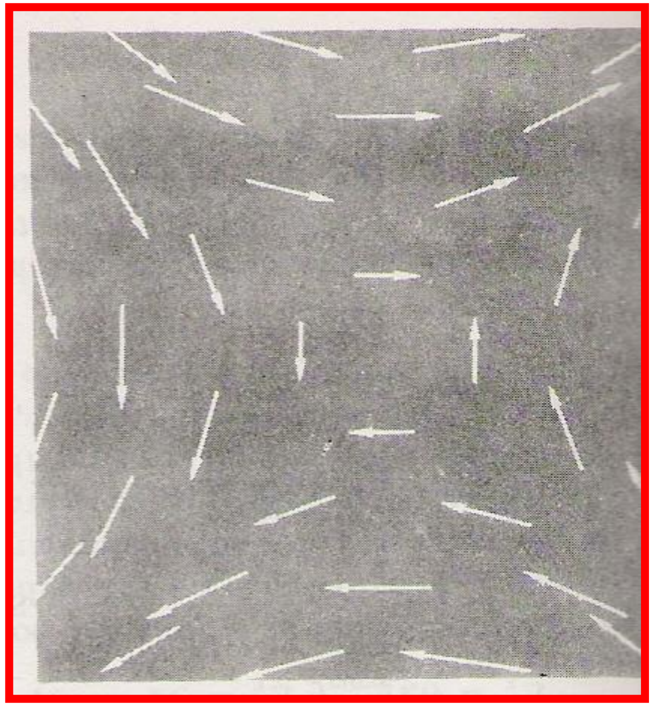
From Berkeley series text book on Electricity and Magnetism

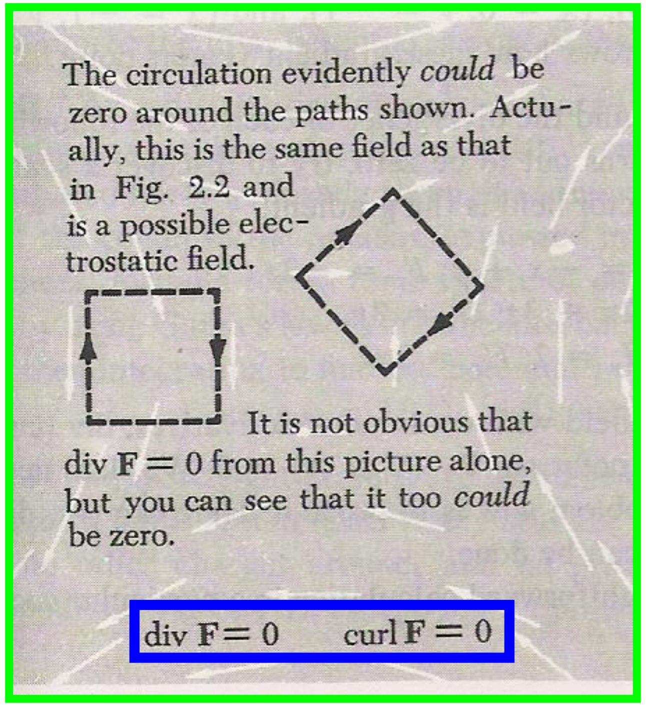




This is a central field. That is, **F** is radial and for given r, its magnitude is constant. Any central field has zero curl; the circulation is zero around the dotted path, and any other path. But the divergence is obviously not zero.

$\text{div}\,\mathbf{F} \neq 0 \qquad \text{curl}\,\mathbf{F} = 0$

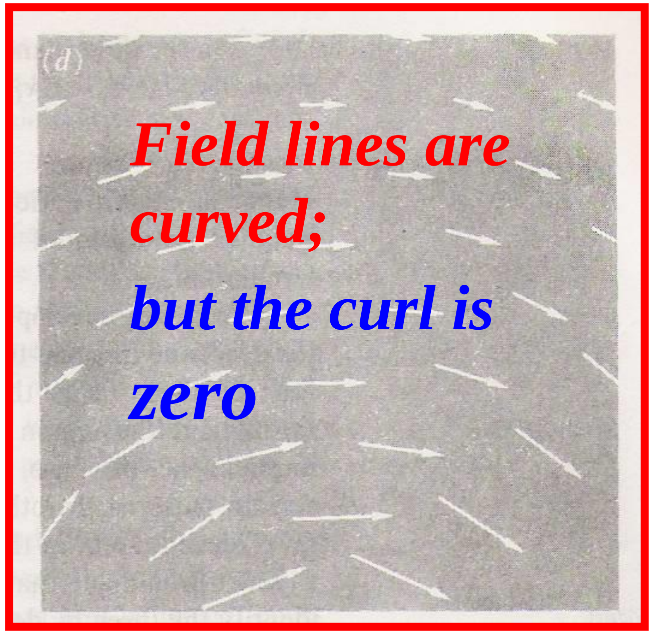


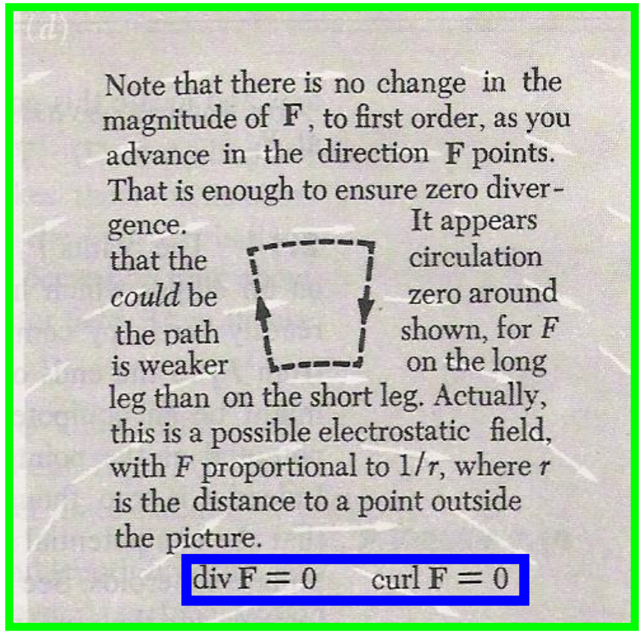

The circulation evidently *could* be zero around the paths shown. Actually, this is the same field as that in Fig. 2.2 and is a possible electrostatic field.

It is not obvious that div **F** = 0 from this picture alone, but you can see that it too *could* be zero.

div **F** = 0 curl **F** = 0

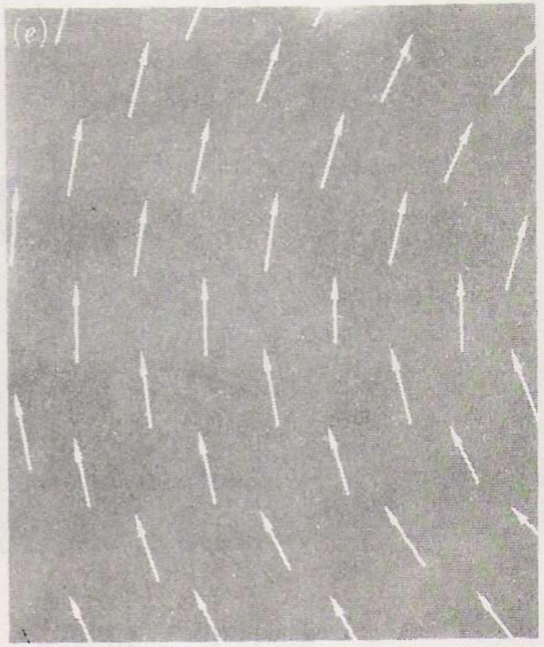
From Berkeley series text book on Electricity and Magnetism



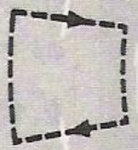

(d)

Note that there is no change in the magnitude of **F**, to first order, as you advance in the direction **F** points. That is enough to ensure zero divergence. It appears that the circulation *could* be zero around the path shown, for *F* is weaker on the long leg than on the short leg. Actually, this is a possible electrostatic field, with *F* proportional to $1/r$, where r is the distance to a point outside the picture.

div **F** = 0 curl **F** = 0

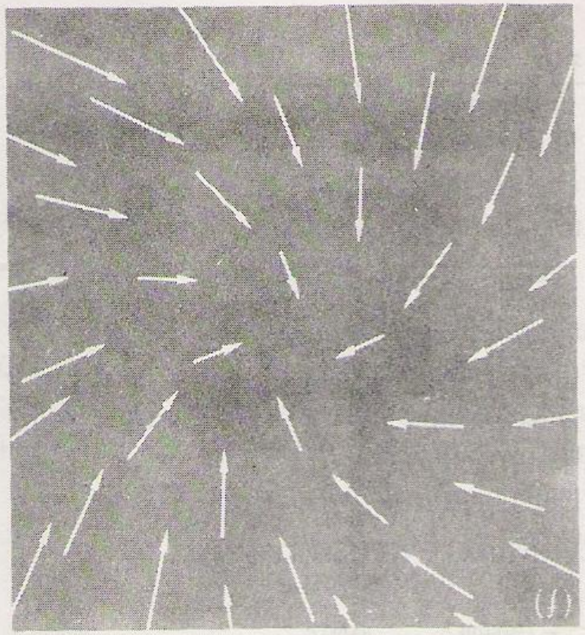


For the same reason as above, we deduce that div **F** is zero. Here the magnitude of *F* is the same everywhere, so the line integral over the long leg of the path shown is not canceled by the integral over the short leg, and the circulation is not zero.



div **F** = 0 curl **F** ≠ 0

From Berkeley series text book on Electricity and Magnetism



Clearly the circulation around the dotted path is not zero. There appears also to be a nonzero divergence, since we see vectors converging toward the center from all directions.

div **F** ≠ 0 curl **F** ≠ 0