Final Exam
PH 5080 - Statistical Physics
Course Instructor: Ashwin Joy

Date: May 1, 2017
Time Duration: 13:00 hrs - 16:00 hrs
Total Marks: 50
Electronic devices such as smart-phones, calculators are not allowed.

List of Formulae

Stirling Approximation: \( N! \approx (N/e)^N \sqrt{2\pi N} \).

Gaussian Integral: \( \int_{-\infty}^{+\infty} dx \ e^{-ax^2} = \sqrt{\pi/ a} \).

Characteristic Function: \( p(k) = \int_{x_{\min}}^{x_{\max}} dx \ e^{-ikx} p(x) \) where \( x \in [x_{\min}, x_{\max}] \).

Moments: \( \langle x^n \rangle = \lim_{k \to 0} \left[ \frac{\partial}{\partial (-ik)} \right]^n p(k) \).

Cumulants: \( \langle x^n \rangle_c = \lim_{k \to 0} \left[ \frac{\partial}{\partial (-ik)} \right]^n \ln p(k) \).

Maxwellian (Velocity Distribution): \( p(v_x, v_y, v_z) = \left( \frac{m}{2\pi k_B T} \right)^{3/2} \exp \left( -\frac{m(v_x^2 + v_y^2 + v_z^2)}{2k_B T} \right) \).

Maxwellian (Speed Distribution): \( p(v) = 4\pi v^2 \left( \frac{m}{2\pi k_B T} \right)^{3/2} \exp \left( -\frac{mv^2}{2k_B T} \right) \).

Gamma Function: \( \Gamma(n) = (n - 1)! = \int_0^{\infty} x^{n-1} e^{-x} \, dx \).

Approximation of Common Constants: \( \ln 2 \approx 0.693, \ln 3 \approx 1.098, \sqrt{2\pi} \approx 2.5 \).
Part I (16 Marks)

Pick the correct answer/s from the choices given next to each question.

1. The function $e^{-(\sqrt{\pi}x)^2}$ with $-\infty < x < +\infty$
   (a) is not a valid probability density function.
   (b) is a probability density function with zero mean and variance $2\pi$.
   (c) is a probability density function with unit mean and variance $(2\pi)^{-1}$.
   (d) is a probability density function with zero mean and variance $(2\pi)^{-1}$.

2. A green coin and a red coin are tossed four times each. Taking the coins to be unbiased, what is the probability of getting two red heads and three green tails?
   (a) $3/32$.
   (b) $5/32$.
   (c) $7/32$.
   (d) $9/32$.

3. Consider a barrel containing one each of the twenty-six letters of the alphabet. The probability of drawing the letters out in exactly the order of the alphabet, A to Z is
   (a) $1/26$.
   (b) $26/2^{26}$.
   (c) $1/2^{26}$.
   (d) $1/(26!)$.

4. An ideal gas with fixed volume and number of particles exists at a thermal equilibrium. Then
   (a) $\langle E \rangle_c = 2.5/\beta$ and $\langle E^2 \rangle_c$ scales as $N^{1/2}$.
   (b) $\langle E \rangle_c = 2.5/\beta$ and $\langle E^2 \rangle_c$ scales as $N^1$.
   (c) $\langle E \rangle_c = 1.5/\beta$ and $\langle E^2 \rangle_c$ scales as $N^{1/2}$.
   (d) $\langle E \rangle_c = 1.5/\beta$ and $\langle E^2 \rangle_c$ scales as $N^1$.

5. For an ideal gas enclosed within impermeable walls at constant pressure and temperature,
   (a) only energy will fluctuate.
   (b) both energy and volume will fluctuate.
   (c) both volume and particle number will fluctuate.
   (d) energy, volume and particle number will fluctuate.

6. If $A$ is a quantity that fluctuates in a canonical ensemble, then ensemble equivalence demands
   (a) $\sqrt{\langle A^2 \rangle_c}/\langle A \rangle_c \sim N^1$.
   (b) $\sqrt{\langle A^2 \rangle_c}/\langle A \rangle_c \sim N^{1/2}$.
   (c) $\langle A^2 \rangle_c/\langle A \rangle_c \sim N^0$.
   (d) $\langle A^2 \rangle_c/\langle A \rangle_c \sim N^{-1/2}$.

7. For a $d$-dimensional Ising model subject to Hamiltonian $H = -J \sum_{\langle ij \rangle} S_i S_j$ and $J > 0$,
   (a) at some $T < T_c \neq 0$ and $d \geq 2$, ferromagnetism may occur.
   (b) at some $T < T_c \neq 0$ and $d \geq 2$, anti-ferromagnetism may occur.
   (c) at some $T < T_c \neq 0$ and $d \geq 2$, both ferro- and anti-ferromagnetism may occur.
   (d) a magnetic phase transition is not possible at any $T$ as magnetic field is zero.

8. For the free energy of a system in canonical ensemble $F = -\beta^{-1}\ln Z$, we can say that
   (a) $F > 0$.
   (b) both $F$ and $(\partial F/\partial T)_V$ must be continuous everywhere.
   (c) $F$ is always continuous but $(\partial F/\partial T)_V$ may be dis-continuous at phase boundary.
   (d) $(\partial^2 F/\partial T^2)_V \geq 0$.  

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Part II (12 Marks)

Do any three of the following questions.

1. In an alpha-particle counting experiment the number of alpha particles is recorded each minute for 50 hours. The total number of particles is 6000. In how many 1-minute intervals would you expect (a) no particles? and (b) exactly 1 particle?

2. Consider a biased coin with probability 1/3 of heads and 2/3 of tails and suppose it is tossed 450 times. What is the probability of getting exactly 300 tails?

Hint: Take the Stirling approximation for large $N$ from the list of formulae.

3. For the continuous random variables $x$ and $y$, write down the expressions for $\langle x^4 \rangle$ and $\langle x^2 y^2 \rangle$ in terms of appropriate cumulants.

4. A random variable $x$ has the probability distribution

$$f(x) = (2\pi)^{-1/2}e^{-x^2/2} \quad (-\infty < x < \infty).$$

If $a = (x_1 + x_2 + \cdots + x_n)/n$, where $x_i$ are all independent, find $p(a)$, the probability distribution of $a$. What is $\langle a \rangle$ and $\langle a^2 \rangle$?

5. A two-level system in the canonical ensemble $(N, T)$ is subject to the Hamiltonian

$$H = \epsilon \sum_{i=1}^{N} n_i, \quad n_i = 0, 1$$

Find the expression for energy $E$ and thus the heat capacity at constant particle number $C_N$.

Part III (12 Marks)

Do any two of the following questions.

1. The random variable $x$ has the probability distribution

$$f(x) = e^{-x} \quad (0 < x < \infty)$$

(a) Find $\langle x^n \rangle$ and $\langle x^n \rangle_c$.

(b) Find $\langle x_1 + x_2 \rangle$ and $\langle x_1 x_2 \rangle$ if $x_1$ and $x_2$ are independent random variables drawn from $f(x)$.

2. An ideal mono-atomic gas is in thermal equilibrium at room temperature $T$ so that the molecular velocity distribution is Maxwellian (refer the list of formulae)

(a) If $v$ denotes the speed of a molecule, calculate $\langle 1/v \rangle$. Compare this with $1/\langle v \rangle$.

(b) Using physical reasoning alone and without explicitly working out any integrals can you find the mean values $\langle v_x \rangle$, $\langle v_x^2 \rangle$, $\langle v^2 v_x \rangle$ and $\langle v_x^2 v_y \rangle$?

3. The Ising model $H = -H \sum_{i=1}^{N} S_i - J \sum_{\langle ij \rangle} S_i S_j$ is used extensively study phase transitions.

(a) Show that the Ising model has time reversal (or $Z_2$) symmetry:

$$F(H, J, T) = F(-H, J, T)$$

(b) Show that under zero external field (or $H = 0$), the Ising model on a square lattice exhibits sub-lattice symmetry:

$$F(0, J, T) = F(0, -J, T)$$

(c) Briefly state the physical consequences of these two symmetries?
Do any one of the following questions.

1. A system of \( N \) distinguishable non-interacting spins in a magnetic field \( H \) is specified by the Hamiltonian

\[
\mathcal{H} = -\sum_{i=1}^{N} S_i \mu H, \quad S_i = \pm 1
\]

where \( S_i \mu \) is the magnetic moment in the direction of the field \( H \).

(a) Determine the behavior of energy \( E \) and entropy \( S \) as the temperature \( T \to 0 \).
(b) Determine second cumulant \( \langle M^2 \rangle_c \) where the instantaneous magnetization \( M = \sum_{i=1}^{N} \mu S_i \).
(c) Determine the heat capacity at constant magnetic field \( C_H \).

2. In a simple one-dimensional “random walk”, steps of unit length are made in the positive or negative \( x \)-direction with equal probability.

(a) Find the probability distribution for the position \( x_n \) after \( n \) steps and find \( \langle x_n \rangle \) and \( \langle x_n^2 \rangle \).
(b) Find the expected number of returns to the origin if a total of \( n \) steps have been taken. Now find the probability that the walker returns to the origin at the \( n^{th} \) step, but not before.

3. Consider an ideal gas in the grand canonical ensemble which are specified by macro-states \((\mu, V, T)\). The unconditional probability of finding \( N \) particles in the system is

\[
p(N) = e^{\beta \mu N} \mathcal{Z}(N, V, T)/\mathcal{Q}(\mu, V, T)
\]

where \( \mathcal{Z}(N, V, T) \) and \( \mathcal{Q}(\mu, V, T) \) are the appropriate partition functions.

(a) Find \( \langle N^2 \rangle_c \). Now get the relative number fluctuations \( \sqrt{\langle N^2 \rangle_c/\langle N \rangle} \) as \( N \to \infty \)?
(b) If the grand potential \( \mathcal{G} = E - TS - \mu N = -\beta^{-1} \ln \mathcal{Q} \), prove that the chemical potential \( \mu = \beta^{-1} \ln (\beta P \lambda^3) \) where \( \lambda = h/\sqrt{2\pi m \beta^{-1}} \).