Lecture 10: Beyond Normal Operators

Ashwin Joy

Department of Physics, IIT Madras, Chennai - 600036

Diagonalization

- Normal operators ($\boldsymbol{A}\boldsymbol{A}^\dagger = \boldsymbol{A}^\dagger\boldsymbol{A}$) are unitarily diagonalizable
- What can we say about other operators?

Theorem

An operator $\boldsymbol{A}$ is diagonalizable if there exists an $\boldsymbol{S}$, such that

$$\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S} = \boldsymbol{A}_{\mathrm{D}} = \begin{bmatrix} \lambda_1 & 0 & \dots \\ 0 & \lambda_2 & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

where the similarity operator $\boldsymbol{S}$ is set up <u>using the eigenkets of $\boldsymbol{A}$</u>

In what follows, we prove this!

Basis Transformation

Consider $\boldsymbol{A}$ in the standard basis, $\mathbb{B} = \{|e_1\rangle, |e_2\rangle, \ldots |e_N\rangle\}$
If there exists an eigenbasis of $\boldsymbol{A}$, $\mathbb{V} = \{|v_2\rangle, |v_2\rangle, \ldots |v_N\rangle\}$

We can construct the transformation operator $\boldsymbol{S}$,

$$\begin{aligned} |v_j\rangle &= \sum_k \boldsymbol{S}_{kj} |e_k\rangle \quad \text{with } |e_k\rangle = \sum_l \boldsymbol{S}^{-1}_{lk} |v_l\rangle \\ \text{or, } \langle e_l | v_j \rangle &= \boldsymbol{S}_{lj} \quad \ldots \mathbb{B} \text{ is orthonormal} \end{aligned}$$

Therefore,

$$\boldsymbol{S} = \begin{bmatrix} \langle e_1|v_1\rangle & \langle e_1|v_2\rangle & \ldots \\ \langle e_2|v_1\rangle & \langle e_2|v_2\rangle & \ldots \\ \vdots & \vdots & \ddots \end{bmatrix} = \underbrace{\begin{bmatrix} \vdots & \vdots & \\ |v_1\rangle & |v_2\rangle & \ldots \\ \vdots & \vdots & \end{bmatrix}}_{\text{eigenkets as columns}}$$

Proof by Geometry

$$
\begin{aligned}
\boldsymbol{AS} &= \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} & \cdots \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \vdots & \vdots & \\ |v_1\rangle & |v_2\rangle & \cdots \\ \vdots & \vdots & \end{bmatrix} \\
&= \begin{bmatrix} \vdots & \vdots & \\ \lambda_1 |v_1\rangle & \lambda_2 |v_2\rangle & \cdots \\ \vdots & \vdots & \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} \vdots & \vdots & \\ |v_1\rangle & |v_2\rangle & \cdots \\ \vdots & \vdots & \end{bmatrix}}_{\boldsymbol{S}} \underbrace{\begin{bmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}}_{\boldsymbol{A}_{\mathrm{D}}}
\end{aligned}
$$

Therefore,

$$
\boxed{\boldsymbol{S}^{-1}\boldsymbol{AS} = \boldsymbol{A}_{\mathrm{D}}}
$$

Proof by Dirac Formalism

$$\begin{aligned}
(\boldsymbol{AS})_{ij} &= \sum_l \boldsymbol{A}_{il}\boldsymbol{S}_{lj} \\
&= \sum_l \langle e_i|\boldsymbol{A}|e_l\rangle \langle e_l|v_j\rangle \\
&= \langle e_i|\boldsymbol{A}|v_j\rangle \quad \ldots \text{closure of } \mathbb{B} \\
&= \langle e_i|v_j\rangle \lambda_j \quad \ldots \boldsymbol{A}\,|v_j\rangle = \lambda_j\,|v_j\rangle \\
&= \boldsymbol{S}_{ij}\lambda_j \\
&= \sum_k \boldsymbol{S}_{ik}\delta_{kj}\lambda_k \\
&= \sum_k \boldsymbol{S}_{ik}\boldsymbol{A}_{\mathrm{D}\,kj} \\
&= (\boldsymbol{SA}_{\mathrm{D}})_{ij}
\end{aligned}$$

yielding,

$$\boldsymbol{AS} = \boldsymbol{SA}_{\mathrm{D}} \implies \boxed{\boldsymbol{S}^{-1}\boldsymbol{AS} = \boldsymbol{A}_{\mathrm{D}}}$$

Simultaneous Diagonalization

For some $\boldsymbol{A}$ and $\boldsymbol{B}$, there exists a similarity operator $\boldsymbol{S}$, such that

$$\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S} = \boldsymbol{A}_{\mathrm{D}} \quad \text{and} \quad \boldsymbol{S}^{-1}\boldsymbol{B}\boldsymbol{S} = \boldsymbol{B}_{\mathrm{D}}$$

if and only if, $[\boldsymbol{A}, \boldsymbol{B}] = 0$

Proof

Left as exercise, as it is similar to the case of Hermitian operators

Problem

Q. Consider the following operator in the standard basis

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & 0.5 \end{bmatrix}$$

Compute the following

1. $\text{Tr}\left(e^{\boldsymbol{A}}\right)$
2. $\det\left(e^{\boldsymbol{A}}\right)$
3. $\boldsymbol{A}^{\infty}$

Solution

From the characteteristic equation of $\boldsymbol{A}$,

$$0 = \det\left(\boldsymbol{A} - a\boldsymbol{I}\right) = (1-a)(0.5-a)$$

we get the eigenvalues, $a = 1, 0.5$

Eigenkets

a = 1

$$\begin{bmatrix}0\\0\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1\\0 & -0.5\end{bmatrix}}_{\boldsymbol{A}-\boldsymbol{I}} \underbrace{\begin{bmatrix}x\\y\end{bmatrix}}_{|v\rangle} = \begin{bmatrix}y\\-0.5y\end{bmatrix}, \quad \text{yielding } x = \text{arbitrary}, y = 0$$

We pick a normalized ket, $|v_1\rangle = \begin{bmatrix}1 & 0\end{bmatrix}^{\mathsf{T}}$

a = 0.5

$$\begin{bmatrix}0\\0\end{bmatrix} = \underbrace{\begin{bmatrix}0.5 & 1\\0 & 0\end{bmatrix}}_{\boldsymbol{A}-0.5\boldsymbol{I}} \underbrace{\begin{bmatrix}x\\y\end{bmatrix}}_{|v\rangle} = \begin{bmatrix}0.5x + y\\0\end{bmatrix}, \quad \text{yielding } x = -2y$$

We pick a normalized ket, $|v_2\rangle = \begin{bmatrix}-2/\sqrt{5} & 1/\sqrt{5}\end{bmatrix}^{\mathsf{T}}$

Diagonalize $\boldsymbol{A}$

Since $|v_1\rangle$ and $|v_2\rangle$ are linearly independent, we have

$$\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S} = \boldsymbol{A}_{\mathrm{D}} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

where

$$\boldsymbol{S} = \begin{bmatrix} 1 & -2/\sqrt{5} \\ 0 & 1/\sqrt{5} \end{bmatrix} \text{ yielding, } \boldsymbol{S}^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & \sqrt{5} \end{bmatrix}$$

Exponentiation to an arbitrary integer ($n > 0$) is super easy!

$$\boldsymbol{A}^n = (\boldsymbol{S}\boldsymbol{A}_{\mathrm{D}}\boldsymbol{S}^{-1})^n = \boldsymbol{S}\boldsymbol{A}_{\mathrm{D}}^n\boldsymbol{S}^{-1} = \boldsymbol{S}\begin{bmatrix} 1 & 0 \\ 0 & 0.5^n \end{bmatrix}\boldsymbol{S}^{-1}$$

Therefore,

$$\boxed{\lim_{n\to\infty} \boldsymbol{A}^n = \boldsymbol{S}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\boldsymbol{S}^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}}$$

Functions $\boldsymbol{A}$

For a function of $\boldsymbol{A}$, say

$$e^{\boldsymbol{A}} = e^{\boldsymbol{S}\boldsymbol{A}_{\mathrm{D}}\boldsymbol{S}^{-1}} = \boldsymbol{S}e^{\boldsymbol{A}_{\mathrm{D}}}\boldsymbol{S}^{-1} = \boldsymbol{S}\begin{bmatrix} e & 0 \\ 0 & \sqrt{e} \end{bmatrix}\boldsymbol{S}^{-1}$$

By the cyclicity of trace,

$$\boxed{\mathrm{Tr}\left(e^{\boldsymbol{A}}\right) = e + \sqrt{e}}$$

By the distributivity of determinant,

$$\boxed{\det\left(e^{\boldsymbol{A}}\right) = e^{3/2}}$$