Lecture 2: Basics of Kets & Operators

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Schwarz Inequality for Kets

Recall from Euclidean (real) vectors, $|\boldsymbol{a}|^2|\boldsymbol{b}|^2 \geq |\boldsymbol{a}\cdot\boldsymbol{b}|^2$
For complex vectors, we have

$$\langle a|a\rangle \langle b|b\rangle \geq |\langle a|b\rangle|^2$$

Proof: For some arbitrary $|a\rangle$ and $|b\rangle$, lets define

$$|g\rangle = |a\rangle + \lambda |b\rangle, \quad \lambda \in \mathbb{C}$$

$$\langle g|g\rangle = \langle a|a\rangle + \lambda \langle a|b\rangle + \lambda^* \langle b|a\rangle + |\lambda|^2 \langle b|b\rangle \geq 0.$$

This is true for all λ, including $\lambda = -\frac{\langle b|a\rangle}{\langle b|b\rangle}$, for which we get

$$\langle a|a\rangle - \frac{|\langle a|b\rangle|^2}{\langle b|b\rangle} \geq 0$$

yielding, $\boxed{\langle a|a\rangle \langle b|b\rangle \geq |\langle a|b\rangle|^2}$

Setting up a Basis

- Basis representation of a vector is not unique!

$$|a\rangle = \underbrace{\alpha\,|e_1\rangle + \beta\,|e_2\rangle}_{\mathbb{B}} = \underbrace{\alpha'\,|e_1'\rangle + \beta'\,|e_2'\rangle}_{\mathbb{B}'} = \ldots \text{infinite ways} \ldots$$

- Basis **need not** be orthogonal. For eg., with

$$|e_1'\rangle = |e_1\rangle\,, \quad |e_2'\rangle = |e_1\rangle + |e_2\rangle$$

the above coefficients become,

$$\alpha' = (\alpha - \beta), \quad \beta' = \beta$$

- Next we show how to set up an orthonormal basis in a given $\mathcal{K}$

Gram Schmidt Orthogonalization

$$\underbrace{\{|v_1\rangle, |v_2\rangle, \ldots |v_N\rangle\}}_{\text{linearly independent}} \xrightarrow{\text{construct}} \underbrace{\{|e_1\rangle, |e_2\rangle, \ldots |e_N\rangle\}}_{\text{orthonormal}}$$

Steps to follow:

1. $\dfrac{|v_1\rangle}{|\,|v_1\rangle\,|} = |e_1\rangle$
2. $\dfrac{|v_2\rangle - \langle e_1|v_2\rangle\,|e_1\rangle}{|\,|v_2\rangle - \langle e_1|v_2\rangle\,|e_1\rangle\,|} = |e_2\rangle$
3. $\dfrac{|v_3\rangle - \langle e_1|v_3\rangle\,|e_1\rangle - \langle e_2|v_3\rangle\,|e_2\rangle}{|\,|v_3\rangle - \langle e_1|v_3\rangle\,|e_1\rangle - \langle e_2|v_3\rangle\,|e_2\rangle\,|} = |e_3\rangle$

$$\vdots$$

Generate all N-orthonormal vectors!

Q. How many distinct orthonormal bases can one generate?

A. Upto a maximum of $N!$ if the initial vectors are not orthogonal.

Operators

An operator acts on a ket (from the left side) to generate a new ket

$$\boldsymbol{X}|a\rangle = |b\rangle$$

Some properties

- $\boldsymbol{X} = \boldsymbol{Y}$, if $\boldsymbol{X}|a\rangle = \boldsymbol{Y}|a\rangle$, for any arbitrary $|a\rangle$
- $\boldsymbol{X} = 0$, if $\boldsymbol{X}|a\rangle = 0$, for any arbitrary $|a\rangle$
- $\boldsymbol{X} + \boldsymbol{Y} = \boldsymbol{Y} + \boldsymbol{X}$, "commutative"
- $\boldsymbol{X} + (\boldsymbol{Y} + \boldsymbol{Z}) = (\boldsymbol{X} + \boldsymbol{Y}) + \boldsymbol{Z}$, "associative"
- $\boldsymbol{X}(c_a|a\rangle + c_b|b\rangle) = c_a\boldsymbol{X}|a\rangle + c_b\boldsymbol{X}|b\rangle$, "linearity"
- $\boldsymbol{X}|a\rangle \xleftrightarrow{\text{DC}} \langle a|\boldsymbol{X}^\dagger$, "$\boldsymbol{X}^\dagger$ is Hermitian adjoint of $\boldsymbol{X}$"

Allowed Products

- $\langle a|\boldsymbol{X}|b\rangle = \langle b|\boldsymbol{X}^\dagger|a\rangle^*$
- $\boldsymbol{XY} \neq \boldsymbol{YX}$ "non-commutative"
- $\boldsymbol{XYZ} = \boldsymbol{X}(\boldsymbol{YZ}) = (\boldsymbol{XY})\boldsymbol{Z}$ "associative"

Can you prove that $(\boldsymbol{XY})^\dagger = \boldsymbol{Y}^\dagger\boldsymbol{X}^\dagger$?

$$\text{We know, } \underbrace{\boldsymbol{XY}}_{\text{operator}} |a\rangle \xleftrightarrow{\text{DC}} \langle a| (\boldsymbol{XY})^\dagger$$

$$\text{but this is } \underbrace{\boldsymbol{X}}_{\text{operator}} \underbrace{\boldsymbol{Y}|a\rangle}_{\text{ket}} \xleftrightarrow{\text{DC}} \underbrace{\langle a|\boldsymbol{Y}^\dagger}_{\text{bra}} \underbrace{\boldsymbol{X}^\dagger}_{\text{operator}}$$

Comparing the two, we establish $(\boldsymbol{XY})^\dagger = \boldsymbol{Y}^\dagger\boldsymbol{X}^\dagger$

Warning on illegal products!

$$\underbrace{|a\rangle\langle b|}_{\text{operator}} \underbrace{|c\rangle}_{\text{ket}} = \underbrace{|a\rangle}_{\text{ket}} \underbrace{\langle b|c\rangle}_{\text{scalar}} = \underbrace{\langle b|c\rangle}_{\text{scalar}} \underbrace{|a\rangle}_{\text{ket}} \neq \underbrace{\langle b|}_{\text{bra}} \underbrace{(|c\rangle|a\rangle)}_{\textbf{illegal}}$$

Closure of Orthonormal Bases

An arbitrary ket in the orthonormal basis $\mathbb{B} = \{|e_1\rangle, |e_2\rangle, \ldots |e_N\rangle\}$

$$|a\rangle = \sum_i a_i |e_i\rangle$$

where the coefficients,

$$a_j = \langle e_j|a\rangle \quad \text{because } \langle e_i|e_j\rangle = 0$$

therefore,

$$|a\rangle = \sum_i \underbrace{\langle e_i|a\rangle}_{a_i} |e_i\rangle = \sum_i |e_i\rangle \langle e_i|a\rangle = \underbrace{\left(\sum_i |e_i\rangle \langle e_i|\right)}_{\text{operator}} |a\rangle$$

yielding us the **closure** of $\mathbb{B}$,

$$\sum_i |e_i\rangle \langle e_i| = \boldsymbol{I} \quad \text{"Identity Operator"}$$

Matrix Representation of Kets & Operators

Consider an arbitrary ket

$$|a\rangle = \boldsymbol{X}|b\rangle = \underbrace{\left(\sum_i |e_i\rangle\langle e_i|\right)}_{\boldsymbol{I}} \boldsymbol{X} \underbrace{\left(\sum_j |e_j\rangle\langle e_j|\right)}_{\boldsymbol{I}} |b\rangle = \sum_{i,j} |e_i\rangle\langle e_i|\boldsymbol{X}|e_j\rangle\, b_j$$

A typical component,

$$a_k = \langle e_k|a\rangle = \sum_j \underbrace{\langle e_k|\boldsymbol{X}|e_j\rangle}_{\text{matrix elements}} b_j$$

Looks like a rule,

$$\underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \end{bmatrix}}_{|a\rangle} = \underbrace{\begin{bmatrix} \langle e_1|\boldsymbol{X}|e_1\rangle & \langle e_1|\boldsymbol{X}|e_2\rangle & \dots \\ \langle e_2|\boldsymbol{X}|e_1\rangle & \langle e_2|\boldsymbol{X}|e_2\rangle & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}}_{\boldsymbol{X}} \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \end{bmatrix}}_{|b\rangle}$$

Representation of Dual Vectors

Continuing our discussion, an arbitrary bra vector

$$\langle a| = \langle b| \boldsymbol{Y} = \langle b| \underbrace{\left(\sum_i |e_i\rangle \langle e_i|\right)}_{\boldsymbol{I}} \boldsymbol{Y} \underbrace{\left(\sum_j |e_j\rangle \langle e_j|\right)}_{\boldsymbol{I}} = \sum_{i,j} b_i^* \langle e_i|\boldsymbol{Y}|e_j\rangle \langle e_j|$$

A typical component,

$$\langle a|e_k\rangle = a_k^* = \sum_i b_i^* \underbrace{\langle e_i|\boldsymbol{Y}|e_k\rangle}_{\text{matrix elements}}$$

Looks like a rule,

$$\underbrace{\begin{bmatrix} a_1^* & a_2^* & \dots \end{bmatrix}}_{\langle a|} = \underbrace{\begin{bmatrix} b_1^* & b_2^* & \dots \end{bmatrix}}_{\langle b|} \underbrace{\begin{bmatrix} \langle e_1|\boldsymbol{Y}|e_1\rangle & \langle e_1|\boldsymbol{Y}|e_2\rangle & \dots \\ \langle e_2|\boldsymbol{Y}|e_1\rangle & \langle e_2|\boldsymbol{Y}|e_2\rangle & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}}_{\boldsymbol{Y}}$$