Lecture 6: Eigenvalues & Eigenvectors

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Eigenvalue Equation

If a linear operator $\boldsymbol{A}$ acts on some $|v\rangle$, such that

$$\boldsymbol{A}|v\rangle = \lambda|v\rangle\,, \quad \lambda \in \mathbb{C}$$

then $|v\rangle$ is called the eigenvector of $\boldsymbol{A}$ with eigenvalue λ

Q. How do we determine λ and $|v\rangle$ for a given operator $\boldsymbol{A}$?
A. Transposing the eigenvalue equation,

$$(\boldsymbol{A} - \lambda\boldsymbol{I})|v\rangle = 0$$

with trivial solutions,

$$\boldsymbol{A} - \lambda\boldsymbol{I} = 0 \quad \text{or} \quad |v\rangle = 0$$

that have no physical utility!

Solving the Eigenvalue Problem

Non trivial solutions for

$$\underbrace{(\boldsymbol{A} - \lambda \boldsymbol{I})}_{\boldsymbol{B}} |v\rangle = 0$$

exist only if $\boldsymbol{B}$ is singular, i.e, det $(\boldsymbol{B}) = 0$

Illustration in 2D space

For det $(\boldsymbol{B}) \neq 0$: $\underbrace{\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}}_{\boldsymbol{B}} \underbrace{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{|v\rangle} = 0 \implies \boxed{v_1 = v_2 = 0}$ "null ket"

For det $(\boldsymbol{B}) =$ 0 : $\underbrace{\begin{bmatrix} \alpha & \beta \\ \gamma & \frac{\beta\gamma}{\alpha} \end{bmatrix}}_{\boldsymbol{B}} \underbrace{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{|v\rangle} = 0 \implies \boxed{v_1 = -\frac{\beta}{\alpha} v_2}$ "family"

Characteristic Equation

If $\boldsymbol{A}$ is an $N \times N$ matrix, then

$$\det\left(\boldsymbol{A} - \lambda \boldsymbol{I}\right) = 0$$

is an N^{th} order polynomial equation in λ with roots, $\lambda_1, \lambda_2 \ldots \lambda_N$

The corresponding eigenvectors are obtained from,

$$\begin{aligned}
\left(\boldsymbol{A} - \boldsymbol{I}\lambda_1\right)|v_1\rangle &= 0 \\
\left(\boldsymbol{A} - \boldsymbol{I}\lambda_2\right)|v_2\rangle &= 0 \\
&\vdots \\
\left(\boldsymbol{A} - \boldsymbol{I}\lambda_N\right)|v_N\rangle &= 0
\end{aligned}$$

Example

The x-component of the spin angular momentum of an electron is denoted by the Pauli spin operator

$$\boldsymbol{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Find the eigenvalues and eigenvectors of $\boldsymbol{\sigma}_x$.
Solution: From the characteristic polynomial,

$$\det\left(\boldsymbol{\sigma}_x - \lambda \boldsymbol{I}\right) = \det\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - 1 = 0, \text{ yielding } \lambda = \pm 1$$

For $\lambda = 1$: $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = 0 \implies \boxed{x_1 = y_1}$ i.e. $|v_1\rangle = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}}$

For $\lambda = -1$: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = 0 \implies \boxed{x_2 = -y_2}$ i.e. $|v_2\rangle = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{1}{\sqrt{2}}$

Eigenvectors can form a Basis

Eigenvectors with distinct eigenvalues are linearly independent

Proof: Let $\boldsymbol{A}$ be a N^2 matrix with eigenvalues $\underbrace{\lambda_1}_{|v_1\rangle}, \underbrace{\lambda_2}_{|v_2\rangle}, \ldots \underbrace{\lambda_N}_{|v_N\rangle}$

First assume that $|v_1\rangle$ and $|v_2\rangle$ are linearly dependent

$$
\begin{aligned}
|v_1\rangle &= \alpha_2 |v_2\rangle && \ldots\ \alpha_2 \neq 0 \\
\lambda_1 |v_1\rangle &= \alpha_2 \lambda_1 |v_2\rangle && \\
\lambda_1 |v_1\rangle &= \alpha_2 \lambda_2 |v_2\rangle && \ldots \text{ from } \boldsymbol{A}|v_1\rangle
\end{aligned}
$$

Subtracting last two, we get

$$0 = (\lambda_2 - \lambda_1)|v_2\rangle \quad \text{not possible, since } \lambda_2 \neq \lambda_1$$

Therefore $|v_1\rangle$ and $|v_2\rangle$ must be linearly independent

Extending to Other Kets ...

Next assume that

$$
\begin{aligned}
|v_3\rangle &= \alpha_1 |v_1\rangle + \alpha_2 |v_2\rangle & \ldots\ \alpha_1 \neq 0 \ \ \& \ \ \alpha_2 \neq 0 \\
\lambda_3 |v_3\rangle &= \lambda_3\alpha_1 |v_1\rangle + \lambda_3\alpha_2 |v_2\rangle & \\
\lambda_3 |v_3\rangle &= \lambda_1\alpha_1 |v_1\rangle + \lambda_2\alpha_2 |v_2\rangle & \ldots\ \text{from } \boldsymbol{A} |v_3\rangle
\end{aligned}
$$

Subtracting last two,

$$0 = \alpha_1(\lambda_3 - \lambda_1) |v1\rangle + \alpha_2(\lambda_3 - \lambda_2) |v_2\rangle$$

This is not possible as $\lambda_{1,2,3}$ are distinct (a contradiction!)
Thus, $|v_1\rangle, |v_2\rangle$ and $|v_3\rangle$ must be <u>linearly independent</u>

⋮

continue and show all $\underbrace{\{|v_1\rangle, |v_2\rangle \ldots |v_N\rangle\}}_{\text{basis}}$ are linearly independent

Normal Operators

Commute with their Hermitian adjoint

$$\underbrace{[\boldsymbol{A}, \boldsymbol{A}^\dagger]}_{\text{commutator}} = \boldsymbol{A}\boldsymbol{A}^\dagger - \boldsymbol{A}^\dagger\boldsymbol{A} = 0$$

Eg. include Hermitian, $\boldsymbol{A} = \boldsymbol{A}^\dagger$ and symmetric, $\boldsymbol{A} = \boldsymbol{A}^T$ operators

Theorem

$\boldsymbol{A}$ and $\boldsymbol{A}^\dagger$ share the eigenvectors but with conjugated eigenvalues

Proof: For a typical eigenket of normal $\boldsymbol{A}$, i.e $|v\rangle$, we have

$$\begin{aligned}
0 &= \langle v|[\boldsymbol{A}, \boldsymbol{A}^\dagger]|v\rangle \\
&= \langle v|(\boldsymbol{A}\boldsymbol{A}^\dagger - \boldsymbol{A}^\dagger\boldsymbol{A})|v\rangle \\
&= \langle v|((\boldsymbol{A} - \lambda\boldsymbol{I})(\boldsymbol{A}^\dagger - \lambda^*\boldsymbol{I}) - (\boldsymbol{A}^\dagger - \lambda^*\boldsymbol{I})(\boldsymbol{A} - \lambda\boldsymbol{I}))|v\rangle \\
&= \langle v|(\boldsymbol{A} - \lambda\boldsymbol{I})(\boldsymbol{A}^\dagger - \lambda^*\boldsymbol{I})|v\rangle - \langle v|(\boldsymbol{A}^\dagger - \lambda^*\boldsymbol{I})(\boldsymbol{A} - \lambda\boldsymbol{I})|v\rangle \\
&= \langle v|(\boldsymbol{A} - \lambda\boldsymbol{I})(\boldsymbol{A}^\dagger - \lambda^*\boldsymbol{I})|v\rangle \quad \ldots (\boldsymbol{A} - \lambda\boldsymbol{I})\,|v\rangle = 0 \\
&= |(\boldsymbol{A}^\dagger - \lambda^*\boldsymbol{I})\,|v\rangle\,|^2 \quad \ldots \text{norm}^2
\end{aligned}$$

Only possible if, $\boxed{(\boldsymbol{A}^\dagger - \lambda^*\boldsymbol{I})\,|v\rangle = 0}$

Normal Operators

Their eigenvectors with distinct eigenvalues are orthogonal

<u>Proof:</u> For any two eigenkets of the normal $\boldsymbol{A}$, say $|v_i\rangle$ and $|v_j\rangle$,

$$\begin{aligned}\langle v_i|\boldsymbol{A}^\dagger\boldsymbol{A}|v_j\rangle &= |\lambda_j|^2\langle v_i|v_j\rangle \\ &= \underbrace{(\boldsymbol{A}|v_i\rangle)^\dagger}_{\langle v_i|\boldsymbol{A}^\dagger} \cdot \boldsymbol{A}|v_j\rangle \\ &= (\lambda_i|v_i\rangle)^\dagger\cdot\lambda_j|v_j\rangle \\ &= \lambda_i^*\lambda_j\langle v_i|v_j\rangle\end{aligned}$$

Subtracting last from first, leads to

$$\lambda_j(\lambda_j^* - \lambda_i^*)\langle v_i|v_j\rangle = 0$$

Since $\lambda_i \neq \lambda_j$, our kets must be orthogonal, i.e, $\langle v_i|v_j\rangle = 0$