Lecture 11: Complex Functions

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Functions of a Complex Variable

For a complex number of the form,

$$z = x + iy$$

we can define a complex valued function,

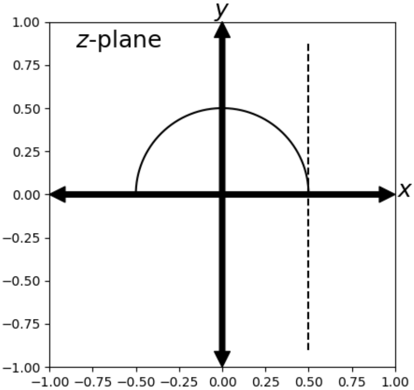
$$f(z) = \underbrace{u(x, y)}_{\text{real}} + i \underbrace{v(x, y)}_{\text{imaginary}}$$

One can generate a mapping of the z-plane onto the f-plane

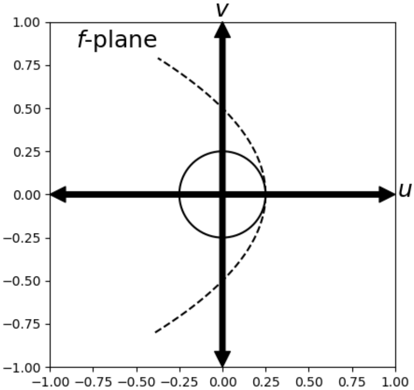
Mapping

Single valued function,

$$f(z) = z^2 = (x+iy)^2 = \underbrace{(x^2 - y^2)}_{u(x,y)} + i\underbrace{2xy}_{v(x,y)} = r^2 e^{i2\theta}$$



Curves, $r = 0.5$, and $x = 0.5$

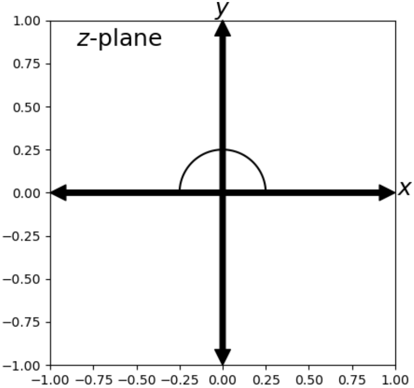


Curves, $r = 0.25$, and $u = 0.25 - v^2$

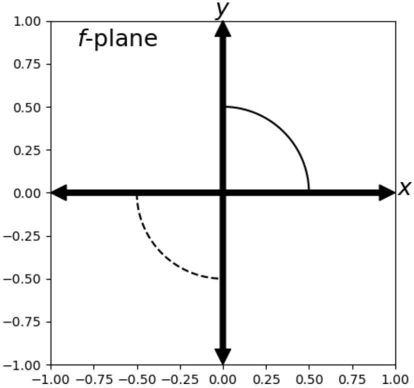
Mapping

Multivalued valued function,

$$f(z) = \sqrt{z} = \begin{cases} \sqrt{r}e^{i\theta/2}, & z = re^{i\theta} \\ \sqrt{r}e^{i(\theta/2+\pi)}, & z = re^{i(\theta+2\pi)} \end{cases}$$



Single Curve, $r = 0.25$



Two curves, $r = 0.5$

Differentiation

A continuous $f(z)$ is differentiable at z_0, if the derivative

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists and is unique.

Examples

1. $f(z) = z^2$

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = 2z$$

   Differentiable everywhere

2. $f(z) = z^*$

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta z^*}{\Delta z} = \lim_{\Delta z \to 0} e^{-2i\zeta} = \begin{cases} +1, & \text{if } \zeta = 0 \ \ (\Delta z \parallel \hat{x}) \\ -1, & \text{if } \zeta = \frac{\pi}{2} \ \ (\Delta z \parallel \hat{y}) \end{cases}$$

   Differentiable nowhere

Cauchy-Riemann Conditions

The continuous function $f(z) = u(x, y) + iv(x, y)$ has a derivative at some $z = x + iy$ if and only if the partial derivatives of u and v exist, and they satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Proof

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \\ &= \lim_{\Delta x \to 0,\, i\Delta y = 0} \frac{f(z + \Delta x) - f(z)}{\Delta x} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= \lim_{\Delta x = 0,\, i\Delta y \to 0} \frac{f(z + i\Delta y) - f(z)}{i\Delta y} = \frac{1}{i}\frac{\partial f}{\partial y} = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \end{aligned}$$

Comparing,

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

vice-versa left as exercise

Analytic Functions

$f(z)$ is analytic in some region $\mathcal{R}$ if it is differentiable inside $\mathcal{R}$

Q. Where is function $f(z) = e^z$ analytic?

$$f(z) = e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos\, y + i \sin\, y)$$
$$\frac{\partial u}{\partial x} = e^x \cos\, y = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -e^x \sin\, y = -\frac{\partial v}{\partial x}$$

$f(z)$ is analytic everywhere except at $z = \infty$ (singularity)
Such functions are called entire

Interesting Example

Q. Examine the case of $f(z) = (x + \alpha y)^2 + 2i(x - \alpha y), \quad \alpha \in \mathbb{R}$

$$u = (x + \alpha y)^2, \quad v = 2(x - \alpha y)$$

$$\frac{\partial u}{\partial x} = 2(x + \alpha y), \quad \frac{\partial v}{\partial y} = -2\alpha$$

$$\frac{\partial u}{\partial y} = 2\alpha(x + \alpha y), \quad -\frac{\partial v}{\partial x} = -2$$

Putting

$$x + \alpha y = -\alpha = -1/\alpha \implies \alpha = \pm 1$$

we see that $f(z)$ is differentiable only on the two straight lines,

$$\boxed{y = -1 - x, \quad y = -1 + x}$$

Therefore, $f(z)$ is analytic nowhere

Polar Version

Cauchy-Riemann conditions have the polar form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Proof

With, $x = r \cos\theta, y = r \sin\theta$ and $\tan\theta = y/x$, we compute

$$\frac{\partial}{\partial x} = \left(\frac{\partial r}{\partial x}\right)\frac{\partial}{\partial r} + \left(\frac{\partial \theta}{\partial x}\right)\frac{\partial}{\partial \theta} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = \left(\frac{\partial r}{\partial y}\right)\frac{\partial}{\partial r} + \left(\frac{\partial \theta}{\partial y}\right)\frac{\partial}{\partial \theta} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

For the Cauchy-Riemann

$$\underbrace{\left(\cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)}_{\frac{\partial}{\partial x}} u = \underbrace{\left(\sin\theta\,\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)}_{\frac{\partial}{\partial y}} v$$

$$\underbrace{\left(\sin\theta\,\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)}_{\frac{\partial}{\partial y}} u = -\underbrace{\left(\cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)}_{\frac{\partial}{\partial x}} v$$

Multiplying the first by cos θ, the second by sin θ, and adding gives

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial\theta}}$$

Multiplying the first by sin θ, the second by $-$cos θ, and adding

$$\boxed{-\frac{1}{r}\frac{\partial u}{\partial\theta} = \frac{\partial v}{\partial r}}$$

For the Derivative

$$
\begin{aligned}
f'(z) = \frac{\mathrm{d}f}{\mathrm{d}z} &= \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \\
&= \left(\cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)u + i\left(\cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)v \\
&= \cos\theta\,\frac{\partial u}{\partial r} + \sin\theta\,\frac{\partial v}{\partial r} + i\cos\theta\,\frac{\partial v}{\partial r} - i\sin\theta\,\frac{\partial u}{\partial r}
\end{aligned}
$$

The last equality follows from CR conditions in polar form. Hence,

$$
\boxed{f'(z) = (\cos\theta - i\sin\theta)\left(\frac{\partial u}{\partial r} + i\,\frac{\partial v}{\partial r}\right) = e^{-i\theta}\,\frac{\partial f}{\partial r}}
$$

Problem

Determine whether the function $f(z) = z^n$ is analytic, for integer n

Solution

$$f(z) = z^n = r^n e^{in\theta} = r^n(\cos\, n\theta + i\, \sin\, n\theta)$$

$$\begin{aligned} \frac{\partial u}{\partial r} &= nr^{n-1}\cos\, n\theta = \frac{1}{r}\frac{\partial v}{\partial \theta} \\ \frac{\partial v}{\partial r} &= nr^{n-1}\sin\, n\theta = -\frac{1}{r}\frac{\partial u}{\partial \theta} \end{aligned}$$

Thus, $f(z)$ is analytic with

$$f'(z) = e^{-i\theta}\,\frac{\partial f}{\partial r} = e^{-i\theta}\, nr^{n-1}e^{in\theta} = nz^{n-1}, \quad n \geq 0$$

For $n < 0$, we need to exclude the singular point, $z = 0$