Lecture 15: Multivalued Functions

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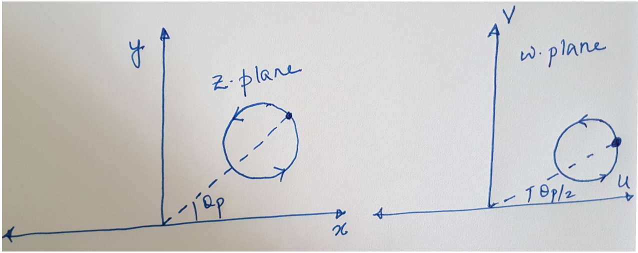
Multivalued Functions

Introduced as the inverse of single valued functions, eg.

$$z = \omega^2$$

Inverting above, yields the simplest multivalued function

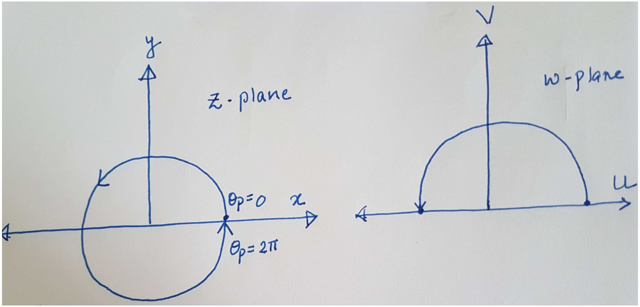
$$\omega = \sqrt{z} = \sqrt{r}\ e^{i\theta_p/2 + in\pi} = \begin{cases} \sqrt{r}\ e^{i\theta_p/2} & \text{(even } n\text{)} \\ -\sqrt{r}\ e^{i\theta_p/2} & \text{(odd } n\text{)} \end{cases}$$



Closed loop away from origin outside returns ω to its original value

Branch Points

Closed loop about origin does not return ω to its original value



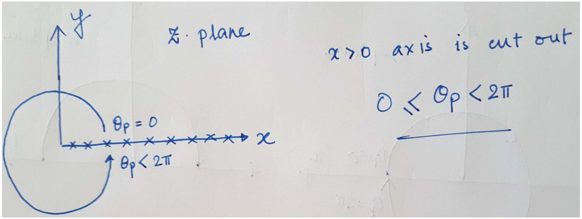
$z = 0$ is therefore a branch point of $\omega = \sqrt{z}$
Another branch point is at $z = \infty$. Take $z = 1/t$ and verify!

Branch cuts

$\omega = \sqrt{z}$ can be made single-valued by a branch cut

Procedure

- Choose an axis joining the branch points $z = 0$ and $z = \infty$
- There are infinite ways to pick $z = \infty$, we choose along $\theta = 0$
- Cut this axis out including $z = 0$ and ∞
- Thus we have fixed the branch at $n = 0$ (principal)



- ω is therefore single valued in this open plane

Complex Logarithm

The function

$$\omega = \ln z = \ln |z| + i\,(\theta_p + 2n\pi) \quad \ldots n \in \mathbb{Z}$$

is infinitely valued! For eg.

$$\ln(-1) = \ln e^{i\,(2n+1)\pi} = i\,(2n+1)\pi$$

Logarithm of positive numbers is taken as single valued

Note that $\omega = \ln z$ has branch points at $z = 0$ and ∞. Removing any ray joining these two branch points, say the $+$ve x-axis is the branch cut

$$\boxed{0 \leq \theta_p < 2\pi}$$

that allows only $n = 0$ thereby making ω single valued.

Handle with Care

Due to multivaluedness, the validity of

$$\boxed{\ln\,(z_1 z_2) = \ln\, z_1 + \ln\, z_2}$$

requires proper specification of branches. For eg., with $z_1 = z_2 = i$,

$$\ln\,(i^2) = \ln\,(i \cdot i) = 2\ln\,(i)$$

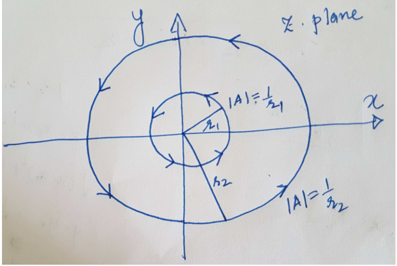
$$\begin{aligned}
m = 0 \quad &: \qquad \ln\,(i^2) = \ln\,((e^{i\pi/2})^2) = 2\ln\,(e^{i\pi/2}) = 2\ln\,(i) \\
m = 1 \quad &: \qquad \ln\,(i^2) = \ln\,((e^{i5\pi/2})^2) = 2\ln\,(e^{i5\pi/2}) = 2\ln\,(i) \\
&\vdots
\end{aligned}$$

Curious Case

Consider the vector field

$$\boldsymbol{A} = \frac{\hat{\theta}}{r} = \frac{-y\hat{x} + x\hat{y}}{r^2}$$

Ex. water flowing down a sink or a magnetic field due to a current carrying wire through origin



Is this vector field conservative?

In what follows, we map this question to a complex variable problem.

$\boldsymbol{A}$ represented by $\Omega(z)$

Notice that away from origin, $\boldsymbol{A}$ is both solenoidal and irrotational

$$\begin{aligned} \nabla \cdot \boldsymbol{A} = 0 \quad &\Longrightarrow \quad \boldsymbol{A} = \nabla \times \psi(x, y)\hat{\boldsymbol{z}} \\ \nabla \times \boldsymbol{A} = 0 \quad &\Longrightarrow \quad \boldsymbol{A} = \nabla \phi(x, y) \end{aligned}$$

The above definitions of $\boldsymbol{A}$ can be used to write

$$\begin{aligned} \phi &= \tan^{-1}(y/x) + C = \arg\,(z) \\ \psi &= -\ln\, r = -\ln\, |z| \end{aligned}$$

Thus, we have the complex function

$$\Omega(z) = \phi + i\psi = \arg\,(z) - i\,\ln\,|z| = -i\,\ln\,(|z|\,e^{i\,\arg\,(z)}) = -i\,\ln\,z$$

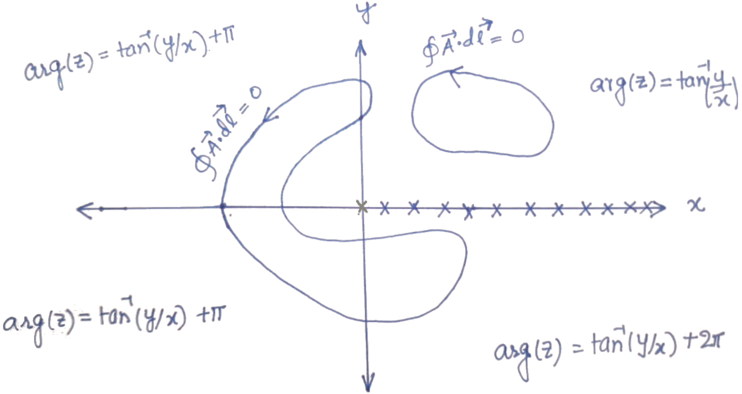
which is single valued/analytic with a branch cut $0 \leq \arg\,(z) < 2\pi$.

$\boldsymbol{A}$ is conservative in the cut plane

Clearly, on any closed path in the cut plane, we have

$$\oint_C \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} = 0$$

$\boldsymbol{A}$ is therefore conservative in this cut plane, see figure below.



Conformal Mapping

Laplace equation in complicated domains can be greatly simplified

Start with a complex potential

$$\Omega(z) = \phi(x, y) + i\,\psi(x, y) \quad \text{analytic in some } \mathcal{R} \in z \text{ plane}$$

Transform to a new variable $\omega = u + i\ v$, via

$$z \equiv F(\omega) \quad \text{analytic in some } \mathcal{R}' \in w \text{ plane}$$

The transformed potential

$$\Omega(z(\omega)) \equiv \Omega(\omega) \quad \text{analytic in same } \mathcal{R}' \in \omega \text{ plane}$$

With complex velocity

$$\frac{\mathrm{d}\Omega}{\mathrm{d}\omega} = \frac{\mathrm{d}\Omega}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}\omega} = \frac{\mathrm{d}\Omega}{\mathrm{d}z}\Big/\frac{\mathrm{d}\omega}{\mathrm{d}z} \quad \ldots \text{ if } \frac{\mathrm{d}\omega}{\mathrm{d}z} \neq 0 \text{ where } \omega = F^{-1}(z)$$

Conformal Mapping

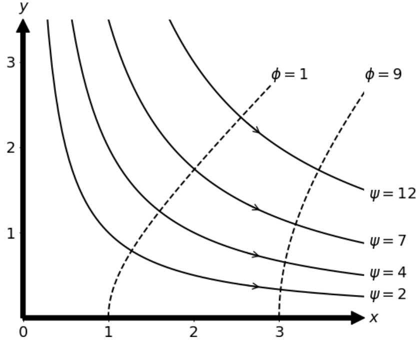
If two curves intersect at a point z_0, then their angle of intersection is preserved by the mapping $z = F(\omega)$ so long as $\left.\frac{d\omega}{dz}\right|_{z_0} \neq 0$

Worked Example

Compute the flow field of an ideal fluid with the complex potential

$$\Omega(z) = z^2 = x^2 - y^2 + i\,2xy$$

Directly reading the potential $\phi = x^2 - y^2$ and streamfunction $\psi = 2xy$, we sketch



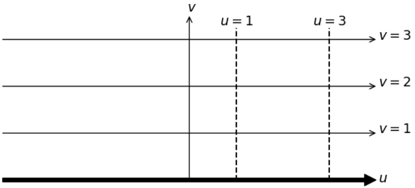
Boundary streamline, $\psi = 0$ at $\theta = 0, \pi/2$
Velocity, $\boldsymbol{v} = 2(x, -y)$ and speed, $|\boldsymbol{v}| = 2r$

The transformation, $z = \sqrt{\omega}$ gives

$$\Omega(z(w)) = \omega = u + i\ v$$

with potential u, and streamfunction v

Uniform straightline flow



Boundary streamline, $v = 0$ at $\theta = 0, \pi$
Velocity, $\boldsymbol{v} = (1, 0)$ and speed, $|\boldsymbol{v}| = 1$
For velocity in z-plane, $\frac{d\Omega}{dz} = \frac{d\Omega}{d\omega}\frac{d\omega}{dz} = 2z$